EXTREMAL PRINCIPLE

On a large, flat field there are n people positioned so that for each person the distances to all the other people are different. Each person holds a water pistol and at a given signal fires at the person who is closest. When n is odd prove that at least one person remains dry. Is this always true when n is even?

Imagine an infinite chess board that contains a positive integer in each square. If the value in each square is equal to the average of the four neighbors to the north, south, east and west prove that the values of all the squares are equal.

There are 2000 points on a circle and each point is given a number which is equal to the average of the two numbers which are its nearest neighbors. Show that all the numbers must be equal.

Place the integers $1, 2, 3, \ldots, n^2$ (without duplication) in any order onto an $n \times n$ chess board with one integer per square. Show that there exists two adjacent squares whose entries differ by at least $n + 1$.

A tennis tournament is held between 20 people and each person plays everyone else. At the end of the tournament each person makes up a list which includes the names of the players he/she beat together with all the names of the players that were beaten by the players that he or she beat. Show that there is someone whose list contains the names of all the other players.

Assume we have a set of n points in the plane and for each pair of points there is a directed line segment from one to the other. Prove that there is a point which can reach every other point by a path of length at most two.

We have previously seen that the greatest number of regions which n lines divides the plane, which we denote by p_n is $\frac{n^2+n+2}{2}$ which happens to equal

$$\binom{n}{2} + \binom{n}{1} + \binom{n}{0}.$$

Try to find a combinatorial/extremal approach to counting the regions that makes this explicit.