Math 30 Introduction to Problem Solving
Final, Friday, March 17, 2000

Directions: Do as many as you can

1. Let a, b, c be integers which satisfy $a^2 + b^2 = c^2$. Prove that abc must be even.

2. (a) Let a_1, a_2, a_3 be three integers and assume that they all have the same parity (are all either even or are all odd). Prove that $16|(a_3 - a_1)(a_2 - a_1)(a_3 - a_2)$. (b) Let a_1, a_2, a_3, a_4 be four integers and assume that they all have the same parity. Prove that
$2^7|(a_4 - a_3)(a_4 - a_2)(a_4 - a_1)(a_3 - a_2)(a_3 - a_1)(a_2 - a_1)$.

3. Let $a_i, i = 1, 2, 3, 4, 5, 6$ be integers. (a) Consider all the 15 differences $a_j - a_i$. Prove that at least two of these are divisible by 3. Can you prove that, in fact, three must be divisible by 3?
(b) Prove that at least one of the differences $a_j - a_i$ is divisible by 5.
(c) Prove that $2^8 3^3 5 | \Pi_{j<i}(a_j - a_i)$.

(d) Is this best possible, that is, is the gcd of all such products $2^8 3^3 5$?

4. (a) Find the shortest path from the point (3,5) to the point (9,3) which touches the $y-$axis.

(b) Find the shortest path from the point (3,5) to the point (9,3) which touches both the $y-$axis and the $x-$axis.

5. (a) Let (V, E) be a graph on n vertices. Prove that either there is an isolated vertex, that is, a vertex without any neighbors, or else there are vertices $x \neq y$ such that x and y have the same number of neighbors.

6. Let (V, E) be a graph on n vertices. Assume that whenever x and y are vertices with the same number of neighbors then they have no common neighbors. Let x be a vertex of valency k with k as large as possible and let $E(x) = \{y_1, y_2, \ldots, y_k\}$ the notation chosen such that $|E(y_j)| \leq |E(y_{j+1})|$. Prove that $|E(y_j)| = j$ for each $j = 1, 2, \ldots, k$.

7. Let $OP_3(n)$ be all ordered triples (A_1, A_2, A_3) of subsets of $\Omega_n = \{1, 2, \ldots, n\}$ such that
I. $A_1 \cup A_2 \cup A_3 = \Omega_n$
II. $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \emptyset$. That is, ordered triples of partitions of Ω_n. Set $op_n(3) = |OP_3(n)|$ the number of such ordered partitions.

(a) How many elements $(A_1, A_2, A_3) \in OP_3(n)$ with $|A_1| = k(\leq n)$?

(b) Use (a) to get a summation formula for $op_3(n)$.

(c) Show that $op_3(n+1) = 3op_3(n)$. (Hint: Think about how you can take an element of $OP_3(n)$ and extend it to an element of $OP_3(n+1)$.) Compare (b) and (c). What explains this identity.

8. Let $a_1, a_2, \ldots, a_n$ be a sequence of positive integers. You may transform these integers by either of the following rules:
I. Multiply some a_i by 2.
II. Subtract 1 from each of the a_i.

(a) Show that it is possible to by a series of such moves to transform the sequence to one with all positive terms and at least one of the terms equal to 1.

(b) Suppose a sequence has one or more ones and it is transformed in the following way: multiply each of the ones by 2 and then subtract one from each of the numbers. Compare the sum of the original sequence with the resulting sequence.

(c) Choose a sequence in the connected component of the original sequence with the sum of the terms as small as possible. Prove each of the terms is one.

9. (a) Let a, b, c be three integers each divisible by 2^n. Prove that there at least one of the three numbers $a+b, a+c, b+c$ is divisible by 2^{n+1}.

(b) Prove that amongst 2^{n+1} integers there is a subset of them with 2^n elements whose sum is divisible by 2^n.

10. We form a sequence of numbers as follows. The first number is $a_1 = 21$. We pick an arbitrary two digit number ab and then $a_2 = 21ab$ (a four digit number). $a_3 = 21abab$ and so on. Prove that no matter what the choice of ab the sequence contains an infinite number of composites.

11. Prove that there are no integer solutions to $x^2 + y^2 = 100003$.

12. Prove that there are no integer solutions to $x^2 - y^2 = 30$.

13. Is it possible to cover a 7×7 chessboard with dominos.

14. Let a be a real number. Then one of the four numbers $a, 2a, 3a, 4a$ is at most $\frac{1}{5}$ from an integer.

15. (a) Let $P_1, \ldots, P_6$ lie on a circle and be the vertices of a regular hexagon. Let the vertices be paired and the chords drawn between the pairs. Prove that two must be of equal length.

(b) Prove the same thing for the eight vertices of a regular octagon.

(c) Prove the same thing for the vertices of a regular $2n-$gon.

16. (a) Let $a, b \geq 0$. Prove that $\frac{a+b}{2} \geq \sqrt{ab}$.

(b) Assume $a, b, c \geq 0$. Prove that $(a+b)(a+c)(b+c) \geq 8abc$.

17. Nine points are given the $\mathbb{R}^3$ and all the 36 lines segments joining them are drawn. They are colored either red or blue. For a vertex x let $R(x)$ be the red edges containing x and $B(x)$ the blue edges containing x. Prove that there is a vertex x such that $|R(x)| \neq 3$.
(b) Assume that there are no red triangles. Prove that every vertex belongs to a blue tetrahedron.

18. At midnight a virus is placed into a colony of 2000 bacteria. Each second each virus destroys one bacterium, after which all the bacteria and viruses divide in two. Prove that eventually all the bacteria will be destroyed and determine the exact time (up to a second) at which this will occur.

19. One hundred black checkers and one hundred red checkers are laid out horizonetally on a set of boxes labelled 1 to 200. Numbers 1 and 200 are black. Prove that there is a number $n < 200$ such that there are an equal number of black and red checkers on squares 1 to n.

20. Given 8 points in the plane. Prove that there exists a line such that 4 points lie on each side of it.

21. The number $abc = a \times 10^2 + b \times 10 + c$ with $a, b, c \in \{0, 1, 2, \ldots, 9\}$ is prime. Can $b^2 - 4ac$ be a perfect square.

22. (a) The points in the plane are colored using three different colors. Prove that it is possible to find a rectangle with all vertices the same color.
(b) The points in the plane are colored using one hundred different colors. Prove that it is possible to find a rectangle with all vertices the same color.

23. Seventeen points, no three collinear, are given in the plane and all the line segments joining them are drawn. The segments are colored red, blue and yellow. Prove that there exists a monochromatic triangle, i.e. a triangle with all three edges the same color.

24. Seven coins are on a table with all heads up. On any move you can turn over four of them. Is it possible to ever get them so that all they are all tails up?

25. Let $a, a+2, a+4, a+6$ be four positive integers in an arithmetic progression with common difference 2. Is it possible for their product to be a perfect square?

26. Define a sequence of integers in the following way: $a_0 = 1$. Suppose $a_0, a_1, \ldots, a_{2^k-1}$ have been defined. Then

$$a_{2^k}, a_{2^k+1}, \ldots, a_{2^{k+1}-1} = 2a_0, 2a_1, \ldots, 2a_{2^k-1}.$$

For example, $a_1 = 2a_0 = 2$. Now $a_0, a_1 = 1, 2$ and hence $a_2, a_3 = 2, 4$ and then $a_4, a_5, a_6, a_7 = 2, 4, 4, 8$ and so on. Prove that $a_n = 2^l$ where there are l ones in the binary expansion of n. For example, $6 = 4 + 2 = 2^2 + 2^1 = 110$. Thus $a_6 = 2^2 = 4$ which is confirmed by the above sequence. As another example, lets try $n = 13$. $13 = 8+4+1 = 2^3+2^2+1 = 1101$ and so we should have $a_{13} = 2^3 = 8$. Extending the sequence to 15 we have 1,2,2,4,2,4,4,8,2,4,4,8,4,8,8,16 and the 14^{th} element in the sequence is, indeed, eight.

27. In Pascals triangle how many numbers are odd in the n^{th} row. Note that in the triangle 1 is the zeroth row and 11 is the first row.

28. Let T be an acute triangle. Inscribe a rectangle R in T with one side along a side of T. Then inscribe a rectangle S in the triangle formed by the side of R opposite the side on the boundary of T, and the other two sides of T, with one along the side of R. For any polygon X, let $A(X)$ denote the area of X. Find the maximum value, or show no maximum exists, of

$$\frac{A(R) + A(S)}{A(T)}$$

where T ranges over all triangles and R, S over all rectangles as above.

29. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying

$$x^4 + 36 \leq 13x^2.$$

30. Inscribe a rectangle of base b and height h in a circle of radius one, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of h do the rectangle and the triangle have the same area?

31. Let R be the region consisting of the points (x, y) of the Cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Sketch the region R and find its area.

32. Let A be an $n \times n$ matrix and $v_1, v_2, \ldots, v_{n+1}$ vectors satisfying

$$Av_i = \lambda_i v_i$$

for some real numbers λ_i (so that v_i is eignevectors with eigenvalues λ_i.) Assume that any n of these vectors are linearly independent. Prove that all the λ_i are equal.