Math 30 Problem Solving
First Problems

1. Fill in the cells of a 3×3 grid with the numbers $1, 2, \ldots, 9$ such that each row, column and diagonal has the same sum. (Free throw).

2. Six points are given in the plane, no three collinear. They are joined by all possible line segments (how many?) and each line segment is colored either red or blue (arbitrarily, you don't get to choose the coloring). Show that there must be a set of three points which span a monochromatic triangle. (Field goal).

3. A domino is a 2×1 rectangle. Can an 8×8 chessboard be covered if the squares A1 and H8 are removed? (Free throw)

4. Let $x_i \in \{\pm 1\}, 1 \leq i \leq n$. Assume that

$$x_1x_2 + x_2x_3 + x_3x_4 + \ldots + x_{n-1}x_n + x_nx_1 = 0.$$

Prove that $4|n$. (Field goal)

5. There are 30 students in a class. Is the following possible:
9 students have 3 friends in class.
11 students have 4 friends in class

6. Find the sum of the infinite series

7. Suppose we fill up a 4×4 grid with all 1's except the (1,2) entry which has a -1. Suppose we are allowed to perform any sequence of the following operations:
R_i change the sign of all entries in the i^{th} row.
C_j change the sign of all entries in the j^{th} column. It is possible the following types of operations to transform the grid into all 1's? (Field goal)

Generalization: Suppose we also allow
PD change all the signs of the entries which are parallel to one of the two diagonals (in particular, you can

8. Given 5 lattice points in the plane prove that you can find a pair amongst them such that the line segment joining them has a third lattice point on its interior (not necessarily from the same set). (Field goal) Generalization: Given 9 lattice points in three space prove that you can find a pair

amongst them such that the line segment joining them has a third lattice point on its interior.

9. Given 12 distinct positive integers prove that there is some pair of them a and b such that either 20 divides a + b or 20 divides a-b. (Field goal)

10. Prove that there must be two distinct powers of 2 such that their difference is divisible by 37. Use this to show that there is a power of 2, 2^e for some e such that $37|2^e - 1$. What about your proof makes use of 37. Can you extend your solution to prove that if n is an odd number then there is a power of 2, 2^e such that $n|2^e - 1$. Can you extend your proof further to the following let m, n be relatively prime integers. Then there is a power of m, m^e such that $n|m^e - 1$? (Beyone the arc).

11. A set of boxes are labeled 1 to 6. How many ways are there to put 20 identical balls into these boxes (it is allowed that some of the boxes may be empty)? (Field goal)

12. A set of boxes are labeled 1 to 6. How many ways are there to put 20 identical balls into these boxes so that none of the boxes is empty? (Field goal)

13. Let n be a natural number. What is the sum of the first consecutive n numbers? (Free throw)

14. Determine which numbers can be written as a sum of (two or more) consecutive natural numbers. Also, describe how one can determine in how many ways a number can written as a sum of consecutive natural numbers (and include an example). For example, 15 = 7 + 8 = 4 + 5 + 6. (Field goal)

15. Let n be a natural number. Determine the sum of the first consecutive odd natural numbers. (Free throw)

16. Determine which numbers can be written as a sum of two or more consecutive odd natural numbers. Also, describe how you can determine all possible ways to write such such a number and provide an example. For example,

$$135 = 43 + 45 + 47 = 7 + 9 + \ldots + 23 = 23 + 25 + 27 + 29 + 31.$$

17. Let $f : \mathbb{N} \to \mathbb{N}$ be a function which satisfies:

$$f(1) = 1, f(2n) = f(n), f(2n+1) = 1 + f(2n)$$

for all $n \in \mathbb{N}$. Find a simple algorithm for determining $f(n)$. (Free throw)

18. Consider a triangle in the plane all of whose vertices are lattice points. Let A denote the area of the triangle, B the number of lattice points on the boudary, and I the number of lattice points in the interior of the triangle. Determine a relationship between A, B and I. (Field goal)

19. Let $OP(n)$ be the number of ordered ways of expressing the natural number n as a sum of natural numbers where the ordering counts. For example:

$$3 = 3 = 2 + 1 = 1 + 2 = 1 + 1 + 1$$

so $OP(3) = 4$ and

$$4 = 4 = 3+1 = 1+3 = 2+2 = 2+1+1 = 1+2+1 = 1+1+2 = 1+1+1+1$$

so $OP(4) = 8$. Determine an explicit formula for $OP(n)$. (Field goal)

20. The Fibonacci sequence is given by $f_1 = f_2 = 1, f_{n+2} = f_n + f_{n+1}$. Prove for each natural number n that

$f_1 + f_3 + \ldots + f_{2n-1} = f_{2n}$. (Free throw) 21. Let f_n be the Fibonacci sequence. Prove that $f_2 + f_4 + \ldots + f_{2n} = f_{2n+1} - 1$. (Free throw)

22. Let f_n be the Fibonacci sequence. Prove that

$$\Sigma_{i=1}^{n} f_i^2 = f_n \times f_{n+1}.$$

(field goal)

23. A group of n people are standing in a circle and are numbered consecutively clockwise from 1 to n. Starting with the person numbered 2 we remove every second person, proceeding clockwise. For example, if $n = 7$ the order in which the people are removed is: 2,4,6,1,3,5,7. Let $l(n)$ be the number of the last person removed, so $l(7) = 7$. Find a method for determining $l(n)$. (field goal)

24. Let p be a prime and $m \in \mathbb{N}, m < p$. Show that there is one and only one $n \in \mathbb{N}, n < p$ so that $p | mn - 1$. (field goal)

25. Prove that $\sqrt{3}$ is irrational. (field goal)

26. Prove that $\log_{10} 2$ is irrational. (field goal)

27. Prove that there exists a, b irrationals such that a^b is rational. (field goal)

28. Let S be a subset or $\mathbb{R}$ which is closed under multiplication. Let T, U be disjoint subsets of S whose union is S and assume that the product of any three (not necessarily distinct) elements of T is in T and similarly for U. Prove that either T is closed under multiplication or that U is closed under multiplication. (beyond the arc)

29. Prove that a set of n elements has 2^n subsets (including the empty set). (free throw)

30. Let x be a real number such that $x + \frac{1}{x}$ is an integer. Prove that $x^k + \frac{1}{x^k}$ is an integer for every natural number k.

For $\{f_n\}$ the Fibonacci sequence prove:

31. $f_n < 2^n$. (free throw)

32. $f_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n - \frac{1-\sqrt{5}}{2})^n]$. (beyond the arc)

34. Consider a $2^{2000} \times 2^{2000}$ square with a single 1×1 square removed. Show that this can be tiled with 2×2 ells. An $m \times n$ ell is a connected collection of squares obtained when and $(n-1) \times (m-1)$ rectangle is removed from an $n \times m$ rectangle. (field goal)

35. Two towns, S and W are connected by a road. At sunrise Aida begins biking from S to W along the road while simultaneously Barabara begins biking from W to S. Each person bikes at a constant speed and they cross pathes at noon. Aida reaches W at 5pm while Barbara reaches S at 11:15pm. When was sunrise? (field goal)

36. A bug is crawling on the coordinate plane from (5,9) to (-15,-7) at constant speed one unit per second except in the second quadrant where it travels at $\frac{1}{2}$ units per second. What path should the bug take to complete its journey in as short a time as possible. (free throw)

37. A bug sits on one corner of a unit cube and wishes to crawl to the diagonally opposite corner. If the bug must remain on the surface of the cube what is the length of a shortest path? (free throw)

38. Let a and b be integers greater than 1 which have no common divisor. Prove that

$$\Sigma_{i=1}^{b-1}[\frac{ai}{b}] = \Sigma j = 1^{a-1}]\frac{bj}{a}]$$

and find the value of this common sum. (field goal)

39. There is set of lockers numbered $1, 2, 3, \ldots, 1000$ and all are unlocked. There are 1000 people standing in a line. The first person then walks by and closes every locker. The second person visits every other locker, starting with 2 and opens each of these lockers. The third person visits every third locker, beginning with 3 and changes its state, that is, locks it if it is unlocked and unlocks it if it is locked. This continues until all 1000 people have gone by the lockers. After the final pass, which lockers are locked?

40. Given a number n describe how to determine how many divisors, $d(n)$ it has. (field goal)

41. Find and prove a formula for the product of all the divisors of an natural number n. For example, for $n = 12$ this is

$$1 \times 2 \times 3 \times 4 \times 6 \times 12 = 1728$$

while for $n = 36$ the result is

$$1 \times 2 \times 3 \times 4 \times 6 \times 9 \times 12 \times 18 \times 36 = 10077696.$$

The formula $d(n)$ might be usefully employed in your formula. (field goal)

42. Consider the following two person game: A collection of pennies are placed one next to another on a straight line. A move consisting of removing one or two consecutive pennies (touching). The last person to remove a penny wins. Is there a winning strategy for either player one or player two? (field goal)

43. Consider the following two person game: Given a 99×99 rectangular grid. Each player, in turn puts a penny in an open square which does not have a common horizontal or vertical edge with an occupied square. Initially the grid is empty. Is there a winning strategy for either player one or two? What if the grid is 100×100? (field goal)

44. An ordinary deck of cards with four aces is shuffled and then the cards are drawn one by one until the first ace appears. On average, how many cards are drawn? (field goal)

45. Given any sequence of n distinct integers we compute its swap number in the following way. Going from left to right, whenever we reach a number which is less than the first number in the sequene we swap its position with the first number in the seqence. The swap number is the total number of swaps. For example, the swap number of 5,6,3,4,7,1,2 is 2 since we first swap 5 and 3 and then 1 and 3. Find the average swap number of the $720 = 6!$ different permutations of the integers 1,2,3,4,5,6. (beyond the arc)

46. Imagine an infinite chess board that contains a positive integer in each square. If the value in each square is equal to the average of the four neighbors to the north, south, east and west prove that the values of all the squares are equal. (free throw)

47. There are 2000 points on a circle and each point is given a number which is equal to the average of the two numbers which are its nearest neighbors. Show that all the numbers must be equal. (free throw)

48. On a large, flat field there are n people positioned so that for each person the distances to all the other people are different. Each person holds a water pistol and at a given signal fires at the person who is closest. When n is odd prove that at least one person remains dry. Is this always true when n is even? (field goal)

49. Place the integers $1, 2, 3, \ldots, n^2$ (without duplication) in any order onto an $n \times n$ chess board with one integer per square. Show that there exists two adjacent squares whose entries differ by at least $n + 1$. (field goal)

50. Show that among any $n + 2$ integers, either there are two whose difference is a multiple of $2n$, or there are two whose sum is divisible by $2n$. (field goal)

51. Chose any $(n+1)$ elements from the set $\{1, 2, \ldots, 2n\}$. Show that this subset must contain two integers which are relatively prime. (free throw)

52. A group of people are seated around a circular table at a restaurant. The food is placed on a circular platform in the center of the table and this platform can rotate. Each person ordered a different entree and it turns out that no one has the correct dish in front of him or her. Show that it is possible to rotate the platform so that at least two people will have the correct entree. (free throw)

53. Show that for any positive integer n, there exists a positive multiple of n which contains only the digets 7 and 0. (field goal)

54. A chess player perpares for a tournament by playing some practice games over a period of eight weeks. She plays at least one game per day, but no more than eleven games per week. Show that there must be a period of consecutive days during which she plays exactly 23 games. (field goal)

55. Are there any natural numbers a, b such that $a^2 - 3b^2 = 8$? (free throw)

56. Assume that p and $8p^2 + 1$ are primes. Find p. (free throw)

57. Let a, b be natural numbers. Can $a^3 + b^3 + 4$ be a perfect cube? (field goal)

58. Find a positive number N such that $N + 1, N + 2, \ldots, N + 99$ are all composite. (field goal)

59. What is the greatest number of regions you can divide the plane into with 3 lines? With 4 lines? With 5 lines? With n lines. (Beyond the arc)