INVARIANT PROBLEMS

Let $x_i \in \{\pm 1\}, 1 \leq i \leq n$. Assume that

$$x_1x_2 + x_2x_3 + x_3x_4 + \ldots + x_{n-1}x_n + x_nx_1 = 0.$$

Prove that $4|n$.

Suppose we fill up a 4×4 grid with all 1's except the (1,2) entry which has a -1. Suppose we are allowed to perform any sequence of the following operations:

R_i change the sign of all entries in the i^{th} row.

C_j change the sign of all entries in the j^{th} column. It is possible the following types of operations to transform the grid into all 1's?

Generalization: Suppose we also allow

PD change all the signs of the entries which are parallel to one of the two diagonals (in particular, you can change the signs of the corners). Is it now possible to transform the grid into all 1's?

Take the unit cube in the first quadrant with vertices (0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1) and (1,1,1). At time zero there is a weight at the origin of one pound. At each second you can choose an edge of the cube and put one pound weights at each of its vertices. Is it possible that at some time the cube is balanced, that is, there is the same weight at each vertex?

The following operations are permitted with the quadratic polynomial $ax^2 + bx + c$: (a) switch a and c, (b) replace x by $x + t$ where t is any real number. By repeating these operations, can you transform $x^2 - x - 2$ into $x^2 - x - 1$?

40 different natural numbers are put up on the blackboard. One can erase any two and replace them by their lcm and gcd. Prove that eventually the numbers will stop changing.

Consider a row of $2n$ squares colored alternately black and white. A legal move consists of chosing a contiguous set of squares (one or more squares but they must be next to each other, no gaps are allowed) and inverting their colors. What is the minimum number of moves necessary to make the row entirely one color?

Is it possible to transform the polynomial $f(x) = x^2 + 4x + 3$ into the polynomial $g(x) = x^2 + 10x + 9$ by a sequence of transformations of the form

$$h(x) \to x^2 h(\frac{1}{x} + 1)$$

and

$$h(x) \to (x-1)^2 h(\frac{1}{x-1})?$$

In each square of an 8×8 chessboard there is an integer. You can make the following moves: (a) Choose a 4×4 square and add one to each integer in the chosen square and (b) choose a 3×3 square and add one to each integer in the chosen square. Can you always get a table with every integer divisible by 2?