NUMBER THEORY PROBLEMS

Given 12 distinct positive integers prove that there is some pair of them a and b such that either 20 divides a + b or 20 divides a-b.

Prove that there must be two distinct powers of 2 such that their difference is divisible by 37. Use this to show that there is a power of 2, 2^e for some e such that $37|2^e - 1$. What about your proof makes use of 37. Can you extend your solution to prove that if n is an odd number then there is a power of 2, 2^e such that $n|2^e - 1$. Can you extend your proof further to the following let m, n be relatively prime integers. Then there is a power of m, m^e such that $n|m^e - 1$?

Determine which numbers can be written as a sum of (two or more) consecutive natural numbers. Also, describe how one can determine in how many ways a number can written as a sum of consecutive natural numbers (and include an example). For example, 15 = 7 + 8 = 4 + 5 + 6.

Let n be a natural number. Determine the sum of the first consecutive odd natural numbers. (Free throw)

Determine which numbers can be written as a sum of two or more consecutive odd natural numbers. Also, describe how you can determine all possible ways to write such such a number and provide an example. For example,

$$135 = 43 + 45 + 47 = 7 + 9 + \ldots + 23 = 23 + 25 + 27 + 29 + 31.$$

The Fibonacci sequence is given by $f_1 = f_2 = 1, f_{n+2} = f_n + f_{n+1}$. Prove for each natural number n that $f_1 + f_3 + \ldots + f_{2n-1} = f_{2n}$.

Let f_n be the Fibonacci sequence. Prove that $f_2 + f_4 + \ldots + f_{2n} = f_{2n+1} - 1$.

Let f_n be the Fibonacci sequence. Prove that

$$\Sigma_{i=1}^n f_i^2 = f_n \times f_{n+1}.$$

Let p be a prime and $m \in \mathbb{N}, m < p$. Show that there is one and only one $n \in \mathbb{N}, n < p$ so that $p|mn - 1$.

Prove that $\sqrt{3}$ is irrational.

Prove that $\log_{10} 2$ is irrational.

There is set of lockers numbered $1, 2, 3, \ldots, 1000$ and all are unlocked. There are 1000 people standing in a line. The first person then walks by and closes every locker. The second person visits every other locker, starting with 2 and opens each of these lockers. The third person visits every third locker, beginning with 3 and changes its state, that is, locks it if it is unlocked and unlocks it if it is locked. This continues until all 1000 people have gone by the lockers. After the final pass, which lockers are locked?

Show that for any positive integer n, there exists a positive multiple of n which contains only the digets 7 and 0.

Are there any natural numbers a, b such that $a^2 - 3b^2 = 8$?

Assume that p and $8p^2 + 1$ are primes. Find p.

Let a, b be natural numbers. Can $a^3 + b^3 + 4$ be a perfect cube?

Find a positive number N such that $N + 1, N + 2, \ldots, N + 99$ are all composite.

Is it possible to use 100 zeros, 100 ones and 100 twos and express (in base 10) a perfect square?

Does there exist a natural number n such that $n^2 + n + 1$ is divisible by 15?

The sum of the digits of the number 2^{1000} is computed and than the sum of the digits of this number and so on until a one digit number is obained. What is that number. (free throw)

Let n be a natural number and $k = [\frac{n}{2}]$ that is, the greatest integer in $\frac{n}{2}$. Determine the following sum

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \ldots \binom{n-k}{k}.$$

Show that there does not exist a quadruple of natural numbers (a, b, c, d) such that

$$a^2 + b^2 = 3(c^2 + d^2).$$

Determine all prime p such that $p^2 + 2$ is also prime.

Prove that among 10 consecutive natural numbers there is at least one number which is relatively prime to each of the others.

Suppose $gcd\{a, 10\} = 1$. Prove that for there is a power of a which when divided by 10^{10} leaves a remainder of 1, i.e. its expression in base 10 ends with 10 zeros and a 1.

Let $f(m, n) = 12^m - 5^n$. Determine $min\{f(m, n) : m, n \in \mathbb{N}\}$.

Prove that there are no integer solutions to

$$15x^2 - 7y^2 = 9.$$

a) Let n be a product of k distinct primes. Determine in how many ways n can be expressed as the difference of two positive (integral) squares? For example, $105 = 3 \times 5 \times 7$ and $105 = 11^2 - 4^2 = 13^2 - 8^2 = 19^2 - 16^2$.

b) Suppose $n = 2^a p_1^{e_1} p_2^{e_2} \dots p_t^{e_t}$ where $a \geq 2$. In how many ways can n be written as a difference of two positive (integral) squares? For example, suppose $n = 4 \times 9 \times 5 = 180$. Then

$$180 = 46^2 - 44^2 = 18^2 - 12^2 = 14^2 - 4^2.$$

Seven natural numbers have the property that the sum of any six is divisible by 5. Prove that each number is divisible by 5.

Let a, b, c, d be integers. Show that the product of the six differences $a - b, a - c, a - d, b - c, b - d, c - d$ is divisible by 12.