Problems for 2/14

1. Let n be a positive integer and define

$$f(n) = 1! + 2! + \dots n!$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n) + Q(n)f(n+1).$$

2. For a positive integer n define $a(n)$ to be equal to the number of ways of expressing n as a sum of 1's and 2's where the order counts. For example,

$$4 = 2+2 = 2+1+1 = 1+2+1 = 1+1+2 = 1+1+1+1$$

so $a(4) = 5$.

For an integer $n > 1$ let $b(n)$ denote the number of ways of writing n as a sum of integers each greater than 1, again assuming that order counts and including n itself. For example,

$$6 = 4+2 = 2+4 = 3+3 = 2+2+2$$

so that $b(6) = 5$. Prove that $a(n) = b(n+2)$ for all positive integers n.

3. Consider a set of $2n$ points in the plane, $n > 1$. Suppose that they are joined by at least n^2+1 line segments. Show that at least one triangle is formed. Show that for each n it is possible to have $2n$ points joined by n^2 line segments without any triangles being formed.

4. Let $\Gamma = (V, E)$ be a graph. Consider all colorings of the vertices of Γ with the colors black and white. A coloring is said to be *aataarataa* if each white vertex has at least as many black neighbors as white and each black vertex has at least as many white neighbors as black. Prove that for any graph there always exists an integrated coloring.

5. Is it possible to transform the polynomial $f(x) = x^2 + 4x + 3$ into the polynomial $g(x) = x^2 + 10x + 9$ by a sequence of transformations of the form

$$h(x) \to x^2 h(\frac{1}{x} + 1)$$

and

$$h(x) \to (x-1)^2 h(\frac{1}{x-1})?$$