Math 30
Problems for 2/16

1) Let $a, b.c$ be distinct real numbers and set

$$f(x) = \frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-a)(x-c)}{(b-c)(b-a)} + \frac{(x-b)(x-c)}{(a-b)(a-c)},$$

a quadratic function. Without expanding prove that for every $x, f(x) = 1$.

2) Let $a_0 = 0, a_1 = 1$ and $a_{n+2} = 2a_{n+1} + a_n$. Prove that $2^k | a_n$ if and only if $2^k | n$.

3) For which values of n are all the binomial coefficients

$$\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}$$

even?

4) How many positive integers n are there such that n is an exact divisor of at least one of the numbers $10^{40}, 20^{30}$.

5) Find all solutions in positive integers to

$$x^2 + y^2 + z^2 = 2xyz.$$