Math 30 Introduction to Problem Solving
Some problems for 2/2

1. We have previously seen that the greatest number of regions which n lines divides the plane, which we denote by p_n is $\frac{n^2+n+2}{2}$ which happens to equal

$$\binom{n}{2} + \binom{n}{1} + \binom{n}{0}.$$

Try to find a combinatorial/extremal approach to counting the regions that makes this explicit.

2. Given n planes in three space let s_n be the greatest number of connected components into which these planes divide the space. Can you find a recursion on n which will allow you to determine a closed formula for s_n? Can you find a combinatorial/extremal argument to determine this number?

3. Suppose Ω is a finite set of point in the plane with the property that the line through any two of them contains as least one other point from Ω. Prove that all the points of Ω are collinear (lie on a single line).

4. Show that there does not exist a quadruple of natural numbers (a, b, c, d) such that

$$a^2 + b^2 = 3(c^2 + d^2).$$

5. Let Ω be a finite set of points in the plane with the property that the triangle formed by any three of them has area at most one. Prove that there is a triangle of area at most four which covers all the points.

6. 40 different natural numbers are put up on the blackboard. One can erase any two and replace them by their lcm and gcd. Prove that eventually the numbers will stop changing.

7. Let Ω be a set of points in the plane with the property that every point in Ω is a midpoint of a line segment joining two points of Ω. Prove that Ω is an infinite set.

8. Let $a_{ij} \in \mathbb{N}$ for each $i, j \in \mathbb{Z}$. Assume that a_{ij} is the average of its four neighbors: $a_{i-1,j}, a_{i+1,j}, a_{i,j-1}$ and $a_{i,j+1}$. Prove that all the a_{ij} are equal.