Math 30
Problems for 2/28

2/28.1) Consider all words of length n made from the letters a, b, c, d. How many have an even number of as?

2/28.2) The set $\Omega_n = \{1, 2, \ldots, n\}$ has $2^n - 1$ nonempty subsets. For each such subset multiply the reciprocal of the members and then add all these numbers up. What is this sum? For example, for $n = 3$ the seven subsets are:

$$\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}.$$

The product of the reciprocals for the respective subsets is: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}$. The sum is:

$$: 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} = 3.$$

2/28.3) Let S be a subset of $\{n \in \mathbb{N} : 10 \leq n \leq 99\}$. Prove that one can find two disjoint nonempty subsets of S with the same sum.

2/28.4) a) Let n be a product of k distinct primes. Determine in how many ways n can be expressed as the difference of two positive (integral) squares? For example, $105 = 3 \times 5 \times 7$ and $105 = 11^2 - 4^2 = 13^2 - 8^2 = 19^2 - 16^2$.

b) Suppose $n = 2^a p_1^{e_1} p_2^{e_2} \ldots p_t^{e_t}$ where $a \geq 2$. In how many ways can n be written as a difference of two positive (integral) squares? For example, suppose $n = 4 \times 9 \times 5 = 180$. Then

$$180 = 46^2 - 44^2 = 18^2 - 12^2 = 14^2 - 4^2.$$