Math 30 Introduction to Problem Solving
More Problems
These problems close on Wednesday, February 10

1. Is it possible to use 100 zeros, 100 ones and 100 twos and express (in base 10) a perfect square? (free throw)

2. Prove that

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots (n-1) \times n = \frac{1}{3}(n-1) \times n \times (n+1)$$

(field goal). Generalize to

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \ldots (n-2) \times (n-1) \times n$$

for beyond the arc.

3. Consider the set $\{1, 2, \ldots, n\}$. How many subsets are there which do not contain consecutive integers? (free throw without proof, field goal with proof)

4. Prove that every natural number can be expressed as a sum of distinct Fibonacci numbers. (field goal)

5. Does there exist a natural number n such that $n^2 + n + 1$ is divisible by 15? (free throw)

6. The sum of the digits of the number 2^{1000} is computed and than the sum of the digits of this number and so on until a one digit number is obained. What is that number. (free throw)

7. How many ways are there to express n as a sum of two numbers where the order of the expression counts? Put another way, in how many ways can you put n balls into 2 boxes if each box must be non-empty?

8. Same as above but as a sum of three numbers.

9. Take the unit cube in the first quadrant with vertices (0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1) and (1,1,1). At time zero there is a weight at the origin of one pound. At each second you can choose an edge of the cube and put one pound weights at each of its vertices. Is it possible that at some time the cube is balanced, that is, there is the same weight at each vertex? (beyond the arc)

10. Investigate some planar graphs and count the number of vertices (V), edges (E), and faces(F) [connected components of the plane left when the edges are deleted]. Find a relationship relating these three numbers. (field goal for the relationship, beyond the arc for a proof).

11. Starting at the point (0,0) in the plane a move consists of going one unit to the right or one unit to the left. How many distinct paths are there to the point (4,6)? (free throw) To the point (m,n)? (field goal)

12. How many rectangles with lattice points as vertices are there contained in the rectangle with vertices (0,0), (n,0), (0,m) and (n,m)? (field goal)

13. Given a convex polygon in how many ways is it possible to divide it into triangles whose vertices are the vertices of the polygon? (beyond the arc)

14. There are n people at a party, $P_1, P_2, \ldots, P_n$. Let a_j be the number of people with whom person j is acquainted (we do not assume that a person is acquainted with themself and that this relation is symmetric: if P_j is acquainted with P_k then P_k is acquainted with P_j. Prove that there are two different people at the party who are acquainted with the same number of people, i.e. there are $j \neq k$ such that $a_j = a_k$. (field goal)

15. Let n be a natural number and $k = [\frac{n}{2}]$ that is, the greatest integer in $\frac{n}{2}$. Determine the following sum

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \ldots \binom{n-k}{k}.$$

(beyond the arc)

16. The integers $1, 2, \ldots, n$ are arranged in order. In one step you may take any four integers and exchange the first with the fourth and the second with the third. Prove that, if $\frac{n(n-1)}{2}$ is even then, by means of such steps you may reach the arrangement $n, n-1, \ldots, 2, 1$. However, if $\frac{n(n-1)}{2}$ is odd then you cannot reach this arrangement. (field goal)

17. The following operations are permitted with the quadratic polynomial $ax^2 + bx + c$: (a) switch a and c, (b) replace x by $x + t$ where t is any real number. By repeating these operations, can you transform $x^2 - x - 2$ into $x^2 - x - 1$?

18. A rectangular floor is covered by 2×2 and 1×4 tiles. One tile get smashed but the only extra tiles are of the other type. Show that the floor cannot be covered by rearranging the tiles.

19. Every point of the plane is colored either red or blue. Show that there is a rectangle with vertices of the same color.

20. A tennis tournament is held between 20 people and each person plays everyone else. At the end of the tournament each person makes up a list which includes the names of the players he/she beat together with all the names of the players that were beaten by the players that he or she beat. Show that there is someone whose list contains the names of all the other players.

21. Assume we have a set of n points in the plane and for each pair of points there is a directed line segment from one to the other. Prove that there is a point which can reach every other point by a path of length at most two.