Project List

1. Consider the folloing method for generating a random permutation of the set $\{1, 2, \ldots, n\}$. A in contains one ball marked 1, two balls marked 2, and so on up to n balls marked n. Balls are drawn from the bin with replacement, and the order in which the numbers are drawn from the bin determins the permutation (if a ball marked with a number that has already appeared is drawn, that draw is ignored). Let E_n denote the expected position of n in the permuation of $\{1, 2, \ldots, n\}$ generated in this manner. Find $\lim_{n\to\infty} E_n/n$. (Note, this method of selecting a permutation is similar to how they choose lottery picks in the NBA.)

2. Get a computer program or calculator to run the following experiment (many times).

   (a) Choose a number between 0 and 1 at random.

   (b) Now choose a second random number and add the first two numbers together.

   (c) Check the sum. If the sum is greater than 1, then this trial succeeded in two tries.

   (d) If the sum is less than 1, then pick another random number and add it to the previous two.

   (e) Check the sum of the three random numbers. If this is greater than 1 then the experiment succeeded in 3 tries, and so on.

   Perform this experiment many times. What is the expected value of the number of tries for the sum to exceed 1 based on your experiments? What should the expected value be? Prove your answer. Note: The number you get has something to do with one of the classical numbers in mathematics.

3. Using either a deck of playing cards, computer, or calculator, perform the following experiment:

   (a) Taking the numbers 1 to 10, randomize them.

   (b) Determine whether any of the numbers are in the "right" place (e.g. the sequence $2, 3, 4, 5, 1, 7, 6, 9, 8, 10$ is a failure since 10 is in the correct position; however, the sequence $2, 3, 4, 5, 6, 7, 8, 9, 10, 1$ is a success.

(c) Perform this experiment many times keeping track of how many times you obtain none of the numbers in the right position.

Determine the experimental probability that for any given randomization none of the numbers are in the right position. Find the probability mathematically. Prove your answer. Note: The number you get has something to do with one of the classical numbers in mathematics.

4. Let $P(t) = (a \cos t, b \sin t)$, where $0 \leq t \leq 2\pi$, describe a point moving along the ellipse $x^2/a^2 + y^2/b^2 = 1$. Let $Q(t)$ be continuous with $Q(0) = (0, b)$ and defined on $0 \leq t \leq 2\pi$ by the requirement tha the lines tangent to the ellipse at $P(t)$ and $Q(t)$ are perpendicular. Let $R(t)$ be the intersection point of the two tangent lines. Find the parametric representations for $Q(t)$ and $R(t)$, and describe the curve traced by the point $R(t)$.

5. Choose a random point (x, y) from the unit square in the plane. (The square with vertices at $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.) What is the likelihood that the integer closest to y/x is even? Estimate this answer via experiment using calculator or computer, and then find it mathematically. Note: The answer involves one of the classical numbers in mathematics.

6. A worm is crawling on an elastic band that is 20 inches long. Each day the worm crawls 1 inch. Each evening, a boy stretches the band 10 inches (moving the worm some too). Does the worm ever reach the end of the band? If so, how long does it take him? If not, why not?

7. Take the unit circle, and choose a chord at random from the unit circle. Find the probability that the length of the chord is longer than the length of a side of an inscribed equilateral triangle. Now find this probability in a second way. Explain the surprising answer.

8. What is the probability that a number chosen at random between 0 and 1 has no 5's in its decimal expansion. Give a geometric description of the set of numbers that have no 5 in their decimal expansion. Now, suppose you remove all of the numbers between 1/3 and 2/3 from the unit interval, and then you remove the middle thirds of the two remaining intervals, and then you remove the middle thirds of the remaining

intervals, etc. How many numbers are left over? Give an algebraic description of the numbers that are left.

9. Let $\overline{AB}$ be the longest side of a convex quadrilateral $ABCD$ inscribed in a circle O. Let $\overline{OE}$ and $\overline{OF}$ be the two radii intersecting $\overline{AB}$ which are perpendicular to the diagonals $\overline{DB}$ and $\overline{AC}$ of $ABCD$. Let E' and F' be the feet of the perpendiculars to $\overline{AB}$ from E and F respectively. Prove that $E'F'$ is the arithmetic mean of $|\overline{AD}|$ and $|\overline{BC}|$.

10. Find good rational approximations for $\sqrt{2}$. What makes an approximation good? Do the same for the square root of 3, e and π. Find methods of determining good approximations for a number a given a.

11. Approximate π by rolling a circle once and measuring the length rolled and the diameter. Now do the same by rolling the circle twice. Do three revolutions, etc. Does your estimate improve? Why or why not?

12. Let ABC be an integer-sided right triangle. Let $\overline{CP}$ and $\overline{CQ}$ be, respectively, the median and the altitude to the hypotenuse $\overline{AB}$. Under what conditions are the sides of triangle CPQ integers?

13. Consider the function $\sin(x)$ on the interval $[0, 2\pi]$. Find good approximations to $\sin(x)$ using only step functions. In particular, what are "good" approximations using one step function, two step functions, four step functions, eight step functions, etc? What makes an approximation good?

14. The smallest pair of positive integers so that the Euclidean algorithm when applied to them takes only one step is $(1, 1)$. Similarly,the smallest pair of positive integers so that the Euclidean algorithm takes only two steps is $(3, 2)$. (Step 1: $3 = 2 \cdot 1 + 1$, Step 2: $2 = 1 \cdot 2 + 0$). Find the smallest pair of positive integers such that the Euclidean algorithm takes three steps, four steps, etc. State and prove a theorem for the case of n steps. Given a pair of two positive integers, approximate a bound for how many steps the Euclidean algorithm will take.

15. Take two points x and y_1 distance 1 apart. Let y_2 be distance 1 from y_1 and at a right angle to $\overline{xy_1}$. Let y_3 be at distance 1 from y_2 and at a right angle from $\overline{xy_2}$, etc. What can you say about the figure created

by connecting the points y_1, y_2, etc.? How does this change as the distance between the y values gets smaller and smaller?

16. Take two points x and y_1 distance 1 apart. Consturct a point y_2 so that the triangle xy_1y_2 has area 1, and so that $\overline{xy_1} \perp \overline{y_1y_2}$. Now find a point y_3 such that xy_2y_3 has area 1 and $\overline{xy_2} \perp \overline{y_2y_3}$. Keep going; what sort of figure do you get? What happens if you make the area smaller and smaller?

17. Pick a point at random from $[0, 1]$. Apply Newton's method on $f(x) = x^3 - x$ to this point. What is the probability that Newton's method on this point will converge to 0? What about 1? What about -1? Find the intervals that lead to 0, the intervals that lead to -1 and the intervals that lead to 1. Does every point lead to one of these three numbers?