Comments and Selected Solutions
Homework II

General Comments for All Assignments:

- Please be neat. If you are writing out problems by hand, please use every other line. To make comments, I need a place to write. Also, please leave sufficient margins.
- Using symbols in a sentence is inappropriate mathematical writing style. Symbols for "there exists," "implies," "such that," etc. should only be used in a symbolic sentence.
- You should make sure that you look at the assignment the first day, so that you are not caught by surprise by lines like: "find someone to interview".
- If you are having difficulties with an assignment, **please see me**. Many of the homework problems are difficult. I expect to see people to discuss some of the homework problems. If you visit my office, I can answer your questions, and work with you to see what you are having difficulties with, and try and specifically address those issues. I teach best in my office, please try and take advantage of this from time to time.
- Please staple your papers whenever possible. If you cannot staple, please just fold a corner over, but do not tear the corner. I will try and staple papers at home if they are not stapled. Torn corners just make things harder to deal with.
- Please do not use torn out spiral notebook paper with the fringes left. This makes it much harder for me to keep the homework neat.

Specific Comments on Assignment 2

- Mathematical proofs are often written so that they are easy to check in a step by step way. That is, each sentence of the proof should be easy to check for truth. When reading a proof, check each sentence and see if it makes sense by itself. After you have done this, go back over the whole proof and try and find what the overall theme is.

- When generalizing a proof, you should check to understand each sentence of the proof to see if it makes sense in the changed context. In particular, if you quote a theorem, you should check to make sure that the hypotheses are still correct. If you make a mathematical statement, you should make sure that the changed statement is correct. Also, you should make sure that the pieces still fit together correctly. Some specific examples from this assignment

  1. On the proof that $\sqrt{7}$ is irrational, one might say: "Suppose $\sqrt{7}$ is rational. Then $\sqrt{7} = \frac{x}{y}$ for some integers x and y, such that either x or y is odd. Now $7y^2 = x^2$ so that 7 divides x^2; by Euclid's Lemma, 7 divides x. Writing $x = 7k$, we have that $7y^2 = (7k)^2 = 49k^2$. Dividing by 7, we obtain $y^2 = 7x^2$, and by Euclid's lemma again, we have 7 divides y. Thus $\frac{x}{y}$ is not in least terms, yielding a contradiction.

     However, in this case, the writer has never assumed that x and y have no common factor, so no contradiction has been reached.

  2. On the proof that $\sqrt{21}$ is irrational, someone might try and apply Euclid's lemma to see that if $x^2 = 21y^2$, then 21 divides x. While it is true that 21 must divide x in this case, it is not the case that Euclid's lemma is what tells it to us, since Euclid's lemma assumes the divisor is **prime**.

- Some mathematics problems are very hard. It is fine with me if you can't solve everything. If you run into a problem on a proof, you should get to where you have the problem, and then note that you are stuck at this stage. At that point, tell me what you would like to be true to continue the proof (in the second case above, you would say something like: "I would like to use Euclid's lemma to say that 21 divides x, but it doesn't apply here. Assuming that result is still true, however, I would ..."

- I realize it is a pain in the neck to rewrite similar proofs. If you truly want to avoid this, then you should prove a preliminary lemma before the assignment starts, which covers all the cases. Otherwise, please rewrite the proof. I find it very difficult to point out where the mistake is, when the proof says, "Simply put 21 in where 7 was previously," or similar to problem #5. I have tried very hard to not have too many problems actually too much alike (I don't always succeed).

- Problem 11 on this set was too difficult. I am sorry about that. This doesn't mean that people didn't get it, simply that I regret having assigned it.

Selected Solutions and Comments on Problems.

1. The 57th digit is 1, and the period is 96. It appears that everyone either has a spreadsheet program or a calculator for this now.

2. Again, it looks like everyone has this set up properly, but a few people ran into a few problems. In general, we say that a terminating fraction has period 1 because the 0s repeat. The table that I have is:

n	period	n	period	n	period	n	period
1	1	2	1	3	1	4	1
5	1	6	1	7	6	8	1
9	1	10	1	11	2	12	1
13	6	14	6	15	1	16	1
17	16	18	1	19	18	20	1
21	6	22	2	23	22	24	1
25	1	26	6	27	3	28	6
29	28	30	1	31	15	32	1
33	2	34	16	35	6	36	1
37	3	38	18	39	6	40	1
41	5	42	6	43	21	44	2
45	1	46	22	47	46	48	1
49	42	50	1	51	16	52	6
53	13	54	3	55	2	56	6
57	18	58	28	59	58	60	1

The decimal expansion for $\frac{1}{17}$ is $.\overline{0588235294117647}$, and the expansion for $\frac{1}{19}$ is $.\overline{052631578947368421}$.

3. In this case, the number k_n is equal to the period of $\frac{1}{n}$ if n is not divisible by 2 or 5. We will prove this later on. If n is divisible by 2 or 5, then k_n does not exist.

5. (Proofs that $\sqrt{7}$ is irrational.)
Proof 1: Suppose $\sqrt{7} = \frac{a}{b}$ with $a, b \in Z$. Using the fundamental theorem of arithmetic, we may assume that 7 does not divide both a and b. Multiplying our original equation by b and squaring both sides, we see that $a^2 = 7b^2$. Thus 7 divides a^2. By Euclid's lemma, 7 divides a. As 7 divides a, it follows that $a = 7t$ for some integer t. Plugging this into the equation $a^2 = 7b^2$, we obtain $(7t)^2 = 7b^2$. Thus $7t^2 = b^2$, and 7 divides b^2. Again by Euclid's Lemma, 7 must divide b. This contradicts that not both of a and b were divisible by 7. Consequently, our initial assumption must have been incorrect, and it follows that $\sqrt{7}$ is irrational.

Proof 2: Suppose $\sqrt{7}$ is rational. Then $\sqrt{7} = \frac{a}{b}$ for some non-negative integers a and b. Define $V = \{q \in N \mid q\sqrt{7} \in Z\}$. As $b \in V$, we have that V is non-empty. By the well-ordering principle, V has a least element (we are assuming N does not contain 0 for this proof). Call this element q. Since $2 < \sqrt{7} < 3$, it follows that $2q < q\sqrt{7} < 3q$. Let $p = q\sqrt{7}$, and note that p is an integer since $q \in V$.

Our goal is to show that $p - 2q \in V$ and $0 < p - 2q < q$ as this would contradict that q is the least element of V. First note that $2q < q\sqrt{7} < 3q$ implies $2q < p < 3q$ so that $0 < p - 2q < q$. Moreover as $q\sqrt{7} = p$,

$$\begin{aligned}(p - 2q)\sqrt{7} &= p\sqrt{7} - 2q\sqrt{7} \\ &= (q\sqrt{7})\sqrt{7} - p \\ &= 7q - 2p.\end{aligned}$$

Thus $(p - 2q)\sqrt{7} \in Z$ and $p - 2q \in N$, implying that $p - 2q \in V$. As $p - 2q < q$, we have contradicted that q is the least element of V. Thus our initial assumption that $\sqrt{7}$ is rational must be wrong. Consequently, $\sqrt{7}$ is irrational.

Proof 3: Suppose $\sqrt{7} = \frac{a}{b}$, where a and b are positive integers. As $2 < \sqrt{7} < 3$, it follows that $0 < \sqrt{7} - 2 < 1$. As a result, $(\sqrt{7} - 2)^n$ can be made as small as possible. (The limit as n goes to infinity is 0.) Note that $Z[\sqrt{7}]$ is a ring and is thus closed under multiplication. Hence for each $n \in N$, $(\sqrt{7} - 2)^n \in Z[\sqrt{7}]$, implying that there exists integers A_n and B_n such that $(\sqrt{7}-2)^n = A_n + B_n\sqrt{7}$. Choosing n sufficiently large so that $|(\sqrt{7}-2)^n| < \frac{1}{b}$, we then have that $|A_n + B_n\sqrt{7}| < \frac{1}{b}$. However,

$$A_n + B_n\sqrt{7} = A_n + B_n\frac{a}{b}$$

$$= \quad \frac{A_n b + B_n a}{b}.$$

As $\sqrt{7} - 2 > 0$, this quantity is positive, and as the numerator is an integer, it follows that $A_n + B_n\sqrt{7} \geq \frac{1}{b}$. This contradicts our earlier statement, and hence our initial assumption must be incorrect. Thus $\sqrt{7}$ is irrational.

6.(Proofs that $\sqrt{21}$ is irrational.)

Proof 1: Suppose $\sqrt{21} = \frac{a}{b}$ with $a, b \in Z$. Using the fundamental theorem of arithmetic, we may assume that 3 does not divide both a and b. Multiplying our original equation by b and squaring both sides, we see that $a^2 = 21b^2$. As 3 divides $21b^2$, 3 divides a^2. By Euclid's lemma, 3 divides a. As 3 divides a, it follows that $a = 3t$ for some integer t. Plugging this into the equation $a^2 = 21b^2$, we obtain $(3t)^2 = 21b^2$. Thus $3t^2 = 7b^2$, and 3 divides $7b^2$. Using either a corollary to Euclid's Lemma or the Fundamental Theorem of Arithmetic, we get that 3 must divide b^2. This contradicts that not both of a and b were divisible by 3. Consequently, our initial assumption must have been incorrect, and it follows that $\sqrt{21}$ is irrational.

Proof 2: Suppose $\sqrt{21}$ is rational. Then $\sqrt{21} = \frac{a}{b}$ for some non-negative integers a and b. Define $V = \{q \in N \mid q\sqrt{21} \in Z\}$. As $b \in V$, we have that V is non-empty. By the well-ordering principle, V has a least element (we are assuming N does not contain 0 for this proof). Call this least element q. Since $4 < \sqrt{21} < 5$, it follows that $4q < q\sqrt{21} < 5q$. Let $p = q\sqrt{21}$, and note that p is an integer since $q \in V$.

Our goal is to show that $p - 4q \in V$ and $0 < p - 4q < q$ as this would contradict that q is the least element of V. First note that $4q < q\sqrt{21} < 5q$ implies $4q < p < 5q$ so that $0 < p - 4q < q$. Moreover as $q\sqrt{21} = p$,

$$\begin{aligned} (p - 4q)\sqrt{21} &= p\sqrt{21} - 4q\sqrt{21} \\ &= (q\sqrt{21})\sqrt{21} - 4p \\ &= 21q - 4p. \end{aligned}$$

Thus $(p - 4q)\sqrt{21} \in Z$ and $p - 4q \in N$, implying that $p - 4q \in V$. As $p - 4q < q$, we have contradicted that q is the least element of V. Thus our initial assumption that $\sqrt{21}$ is rational must be wrong. Consequently, $\sqrt{21}$ is irrational.

Proof 3: Suppose $\sqrt{21} = \frac{a}{b}$, where a and b are positive integers. As $4 < \sqrt{21} < 5$, it follows that $0 < \sqrt{21} - 4 < 1$. As a result, $(\sqrt{21} - 4)^n$ can be made as small as possible. (The limit as n goes to infinity is 0.) Note that $Z[\sqrt{21}]$ is a ring and is thus closed under multiplication. Hence for each $n \in N$, $(\sqrt{21}-4)^n \in Z[\sqrt{21}]$, implying that there exists integers A_n and B_n such that $(\sqrt{21}-4)^n = A_n + B_n\sqrt{7}$. Choosing n sufficiently large so that $(\sqrt{21}-4)^n < \frac{1}{b}$, we then have that $|A_n + B_n\sqrt{21}| < \frac{1}{b}$. However,

$$\begin{aligned} A_n + B_n\sqrt{21} &= A_n + B_n\frac{a}{b} \\ &= \frac{A_n b + B_n a}{b}. \end{aligned}$$

As $\sqrt{21}-4 > 0$, this quantity is positive, and as the numerator is an integer, it follows that $A_n + B_n\sqrt{21} \geq \frac{1}{b}$. This contradicts our earlier statement, and hence our initial assumption must be incorrect. Thus $\sqrt{21}$ is irrational.

7. Suppose $\sqrt{8}$ is rational. As the product of two rational numbers is rational, it follows that $\frac{1}{2}\sqrt{8}$ is also irrational. However, $\frac{1}{2}\sqrt{8} = \sqrt{2}$, which we know is irrational. Consequently, our original assumption must have been false, and $\sqrt{8}$ is irrational.

11. The most precise conjecture we could make is:
If $n \in N$ is divisibly by neither 2 nor 5, then the least positive integer k_n such that n divides $10^{k_n} - 1$ is the period of $\frac{1}{n}$.
Proof: Let us first assume that t is the period of $\frac{1}{n}$. By the text, it follows that $\frac{1}{n} = \frac{\alpha}{10^t - 1}$, where α is an integer (note, this is where we are using that neither 2 nor 5 divides n). Thus, cross-multiplying yields that

$$10^t - 1 = \alpha \cdot n.$$

Consequently, n divides $10^t - 1$. It remains to check that t is the least positive integer k_n such that n divides $10^{k_n - 1}$.

Suppose n divides $10^s - 1$ for some positive integer s, and let q_i denote the ith digit of the decimal expansion for $\frac{1}{n}$ and r_i denote the ith remainder in the algorithm for finding the decimal expansion. That is,

$$\begin{aligned} 1 &= n \cdot q_0 + r_1 \\ 10r_1 &= n \cdot q_1 + r_2 \\ 10r_2 &= n \cdot q_2 + r_3 \end{aligned}$$

and so on, where $0 < r_i < n$. We claim that r_i is the remainder of 10^{i-1} upon division by n, for $i = 1, 2, \ldots$. We prove this by induction. For $i = 1$, this is the definition of r_1. Fix $k > 1$ and suppose that r_{k-1} is the remainder of 10^{k-2} upon division by n. At this point, we consider our algorithm modulo n. Since r_{k-1} is the remainder of 10^{k-2} upon division by n, $10r_{k-1} \equiv 10^{k-1}$ modulo n. But the kth equation states $10r_{k-1} \equiv 0 + r_k$ modulo n. Consequently, $10^{k-1} \equiv r_k$ modulo n. Since $0 \leq r_k < n$, it follows that r_k is the remainder of 10^{k-1} upon division by n as claimed. The claim now follows by the principle of mathematical induction.

The above claim implies that $r_{s+1} = 1$. Thus, the digit q_{s+1} is the same as the digit q_1. But this together with $r_{s+1} = 1$ implies that $r_{s+2} = r_2$. Continuing inductively, we have that $q_{s+k} = q_k$ for all k, and thus the period is less than or equal to s.

Thus the period of $\frac{1}{n}$ must be the least k_n such that n divides $10^{k_n} - 1$.

12. *The fraction $\frac{a}{b}$ written in lowest terms has a terminating decimal expansion if and only if b divides 10^k for some non-negative integer k.*
Proof: If $\frac{a}{b}$ has a terminating decimal, then $\frac{a}{b} = \frac{\alpha}{10^k}$ for some non-negative integer k. Cross-multiplying, we have $a \cdot 10^k = b \cdot \alpha$. Thus, if $\frac{a}{b}$ is in lowest terms, we have $\gcd(a, b) = 1$, and a corollary to Euclid's Lemma implies that b divides 10^k. Conversely, if b divides 10^k, then $10^k = b \cdot x$ for some integer x, and then $\frac{a}{b} = \frac{ax}{10^k}$, so that $\frac{a}{b}$ must have a terminating decimal.

14. (Proofs that $\sqrt{28}$ is irrational.)
Proof 1: Suppose $\sqrt{28} = \frac{a}{b}$ with $a, b \in Z$. Using the fundamental theorem of arithmetic, we may assume that 7 does not divide both a and b. Multiplying our original equation by b and squaring both sides, we see that $a^2 = 28b^2$. Thus 7 divides a^2. By Euclid's lemma, 7 divides a. As 7 divides a, it follows that $a = 7t$ for some integer t. Plugging this into the equation $a^2 = 28b^2$, we obtain $(7t)^2 = 28b^2$. Thus $7t^2 = 4b^2$, and 7 divides b^2. By a corollary to Euclid's Lemma, 7 must divide b. This contradicts that not both of a and b were divisible by 7. Consequently, our initial assumption must have been incorrect, and it follows that $\sqrt{28}$ is irrational.

Proof 2: Suppose $\sqrt{28}$ is rational. Then $\sqrt{28} = \frac{a}{b}$ for some non-negative integers a and b. Define $V = \{q \in N \mid q\sqrt{28} \in Z\}$. As $b \in V$, we have that V is non-empty. By the well-ordering principle, V has a least element (we are assuming N does not contain 0 for this proof). Call this element q. Since $5 < \sqrt{28} < 6$, it follows that $5q < q\sqrt{28} < 6q$. Let $p = q\sqrt{28}$, and note that p is an integer since $q \in V$.

Our goal is to show that $p - 5q \in V$ and $0 < p - 5q < q$ as this would contradict that q is the least element of V. First note that $5q < q\sqrt{7} < 6q$ implies $5q < p < 6q$ so that $0 < p - 5q < q$. Moreover as $q\sqrt{28} = p$,

$$\begin{aligned}(p - 5q)\sqrt{28} &= p\sqrt{28} - 5q\sqrt{28} \\ &= (q\sqrt{28})\sqrt{28} - 5p \\ &= 28q - 5p.\end{aligned}$$

Thus $(p - 5q)\sqrt{28} \in Z$ and $p - 5q \in N$, implying that $p - 5q \in V$. As $p - 5q < q$, we have contradicted that q is the least element of V. Thus our initial assumption that $\sqrt{28}$ is rational must be wrong. Consequently, $\sqrt{28}$ is irrational.

Proof 3: Suppose $\sqrt{28} = \frac{a}{b}$, where a and b are positive integers. As $5 < \sqrt{28} < 6$, it follows that $0 < \sqrt{28} - 5 < 1$. As a result, $(\sqrt{28} - 5)^n$ can be made as small as possible. (The limit as n goes to infinity is 0.) Note that $Z[\sqrt{28}]$ is a ring and is thus closed under multiplication. Hence for each $n \in N$, $(\sqrt{28} - 5)^n \in Z[\sqrt{28}]$, implying that there exists integers A_n and B_n such that $(\sqrt{28} - 5)^n = A_n + B_n\sqrt{28}$. Choosing n sufficiently large so that $|(\sqrt{28} - 5)^n| < \frac{1}{b}$, we then have that $|A_n + B_n\sqrt{28}| < \frac{1}{b}$. However,

$$\begin{aligned}A_n + B_n\sqrt{28} &= A_n + B_n\frac{a}{b} \\ &= \frac{A_nb + B_na}{b}.\end{aligned}$$

As $\sqrt{28} - 5 > 0$, this quantity is positive, and as the numerator is an integer, it follows that $A_n + B_n\sqrt{28} \geq \frac{1}{b}$. This contradicts our earlier statement, and hence our initial assumption must be incorrect. Thus $\sqrt{28}$ is irrational.

Problem II: Suppose 2/3 of the men in a particular town are married, and 3/5 of the women are married. Let K denote the number of marriages, M denote the number of men in the town, and W denote the number of women. Then

$$\begin{aligned}\frac{2}{3}M &= K \\ \frac{3}{5}W &= K.\end{aligned}$$

Clearing denominators, we obtain that

$$\begin{aligned} 2M &= 3K \\ 3W &= 5K. \end{aligned}$$

Consequently, K must be divisible by 2 and 3, and hence must be divisible by 6. Letting $K = 6t$, we have that $M = 9t$ and $W = 10t$. Since the number of married people in the town is $2K$, and the number of people in the town is $M + W$, it follows that the fraction of married people in the town is

$$\frac{2 \cdot 6t}{9t + 10t} = 1219.$$

The key step in this proof is recognizing that the number of marriages is what you need to count. (Equivalently, one might say the key is recognizing that $\frac{2}{3}M = \frac{3}{5}W$, although then the algebra can be a little less pleasant.) The difficulty that many people face with this problem is that they are scared of fractions and don't like thinking about what they represent. A very common reaction of students to all word problems is to find the numbers in the problem and try and manipulate them. A second problem faced is that one needs to either put in a supplemental variable (the number of marriages), or be able to manipulate fractions to find the appropriate equivalent fractions to deal with. In this case, to solve the problem, purely algebraically, you have to manipulate some ugly looking terms like $\frac{(2/3)M+(2/3)M}{M+(10/9)M}$, which requires adding and dividing fractions. Students mostly do these tasks by rote memorization of algorithms, which means that when they need to do them a year later, they will often make mistakes.

An interesting way to teach this problem is to have students try and actually create a town having these proportions. Fairly quickly, they will see the need for a common multiple of 2 and 3 to be the number of marriages.