Challenges of Seeing the Math Within:
Issues related to finding, using, and communicating with Rule of 3 representations

There has been a consistent call for teachers to become more knowledgeable about mathematics. Using the Rule of 3 as a way of thinking about instruction has pushed me to more deeply explore the range of mathematics within the problems that I assign and to begin perceiving the relative strengths of different representation in relation to particular problems.

At the beginning of the year I worked through all of the math menu problems (read more about math menus at http://gallery.carnegiefoundation.org/tboerst/align_tasks.htm) from my first unit looking for ways in which numerical, graphic, and algebraic representations could be used in solving and discussing important mathematical issues. It did not take long to realize that a number of the problems would be difficult to represent in certain ways and/or that I would have trouble even conceptualizing the connection between some of the problems and certain representations[i]. It was not initially apparent whether this was: a reason to retool the problems that were on the menus; an instance when it would be necessary to marshal other resources to determine the representations; or the first sign that the Rule of 3 was not going to be consistently useful in working on fifth grade mathematics. It was however an initial hurdle to enacting instruction using the Rule of 3 and a reason to question the extent to which such instruction would advance student learning.

In the following pages I will describe, illustrate, and analyze the work that I did with students and on my own to find the mathematical representations related to fifth grade problem solving tasks. I close by pulling together the pedagogical insights gained through Rule of 3 guided exploration of the math within.

A warning before proceeding- My scholarship in this area is intended to be generative, not definitive. The mathematical ideas stated here are **provisional** at best as there are still many things to be learned about representations that would enrich, extend, or possibly even contradict the statements made here. It is very likely that those reading this work will think of alternative representations not shared here.

Searching for Representations

At first I cast about, searching to see if I could find multiple representations for the problems on the math menu. I will share examples of some early frustrations I experienced in that work. Discussion of the search for representations will rely heavily upon examples of the representations that my students and I developed while solving problems.

Example 1: Finding Them
I start with an example of a problem drawn from the first math menu of the year (see menu samples at http://gallery.carnegiefoundation.org/tboerst/menu_examples_A.pdf) that can easily incorporate the use of numerical, graphic, and algebraic representations. The examples of representations below are ones that students typically share (in writing or verbally) in relation to this problem.

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The menu problem:

Here is a special pattern of numbers that is called "Pascal's Triangle."

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

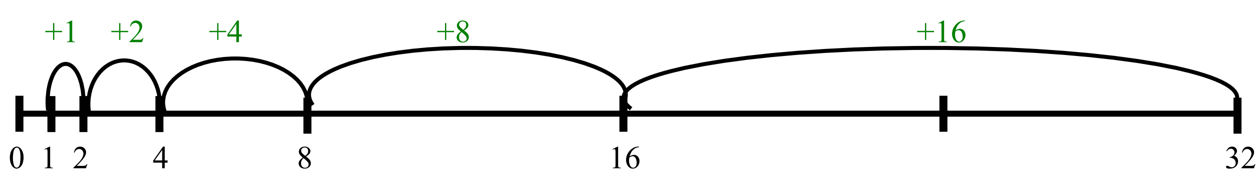
A) Explore a few of the many patterns in this triangle. If you added the numbers in each row (going sideways) what are the sums? Explain the pattern that these sums follow.
B) Try to write the numbers that would be written in the next two rows of the triangle. How did you figure out which numbers went in these rows?

Numerical work related to the problem:

The sums of the rows (going horizontally) are:
1, 1+1= 2, 1+2+1= 4, 1+3+3+1= 8, 1+4+6+4+1= 16, 1+5+10+10+5+1= 32

Graphic work related to the problem:

This also shows the sums of the rows (going horizontally). Graphic representations often document the sums and their growth. Interestingly students sometimes convey their ideas about the growth of the sums orally or with gestures (showing with hands or fingers growth in terms of distance) and then make number lines, graphs, or even equations to capture the same ideas on paper:



Algebraic work related to the problem:

The numbers in any row relate to the numbers in the row under them in the following way (Students say something akin to the following generalized rule, without using variables at this point in the year):

A + B=
C

So for instance in horizontal row five of the triangle the numbers 4 and 6 have the number 10 below/directly down from the space between 4 and 6. This pattern holds true in any place on the triangle (except for the first row).

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Discussion

There are few important things to consider in relation to this problem. First, each representation has value in completing the problem. The algebraic representation is required to communicate about the general patterns and to continue the patterns. Numerical representation is also required as the first part of the problem directs students to find the sums of each row. While a bit more obscure, graphic representations or at least spatial thinking will provide a strong tool for seeking patterns and determining the numbers that should be written the two additional rows of the triangle. This occurs because the triangle is itself a spatial arrangement, so when students undertake work on part B many of them will use graphic representations to look for patterns by dissecting the triangle horizontally, vertically, and diagonally. A second point that must be accounted for is the degree of access that students will have to representations that have the potential to help them in solving problems and communicating their work to others. In this case most students do have easy access to such representations, meaning that they can produce, share, and understand them. A third and related point that merits consideration is the different linguistic guises of representational work. The initial expression of algebraic and graphic representations may be oral and only in some cases or with support might those representations become written. Valuing only written expression of the representations may skew a teacher's perceptions of the extent to which students are making use of the different representations, particularly at early stages of the school year. Furthermore, there are situations where students may not be ready to produce or interpret written representations, but will be active in making and responding to verbal work related to those categories of representations.

Example Set 2: Trouble Finding Them

It becomes evident when carefully examining the two collections of menu examples (see menu samples at http://gallery.carnegiefoundation.org/tboerst/menu_examples_A.pdf and http://gallery.carnegiefoundation.org/tboerst/menu_examples_B.pdf) that there are some math problems where the *utility* of various representations is dubious and where some representations seem to be *out of reach* for most students (or at least out of reach for me pedagogically). Yes, most problems have connections to the Rule of 3 representations, but in some cases the representations do not appear to give much leverage or are quite an intellectual distance from the problem under consideration. Below I will share problem solving examples from menus in a effort to identify some of the representational challenges I faced.

Algebraic Issue:

The menu problem

> Three students work to earn extra money. The first student is paid $6.00 a day for 10 days of work. The second student is paid $.10 the first day, $.20 the second day, $.30 the third day, and so on … for 10 days. The third person is paid one cent the first day, two cents the second day, 4 cents the third day, 8 cents the fourth day,…for a total of 10 days. What is the best way to be paid? Support your answer with math work and explanations.

Representations related to the third person's pattern of payment

Numerically: .01 + .02 + .04 +. 08 + .16 + .32 + .64 + 1.28 + 2.56 + 5.12= $10.23

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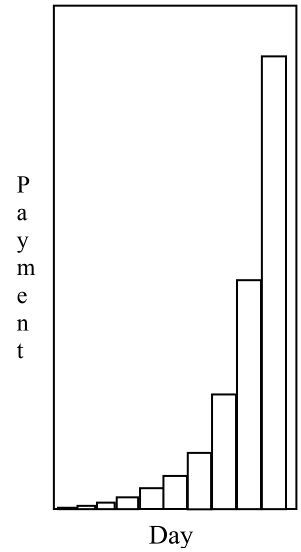
Graphically:

Day	Payment	Total
1	.01	.01
2	.02	.03
3	.04	.07
4	.08	.15
5	.16	.31
6	.32	.63
7	.64	1.27
8	1.28	2.55
9	2.56	5.11
10	5.12	10.23



Algebraically: Daily earnings[a]: $2^{n-1} \times .01$ Where n = number of days
Daily earnings[b]: 2 x b Where b= daily earnings on the previous day
Cumulative earnings: 2d - .01 Where d = daily earning

Discussion

Students typically use numerical representations to solve this problem. They also tend to use tables (graphic representations) to keep track the amount earned each day. While it was in doubt at first, students could see in the graph that the third pattern of payment looked bad at first, but grew very quickly compared to the other methods of payment. This was most strikingly represented on the bar graphs where student could not only compare actual daily amounts but also visually see the rates of growth. During class discussion it didn't take long to see that method C would quickly become the best overall way to be paid if the time period in the problem had been just a bit longer. The algebraic element of this pattern of payment can be quite advanced (daily earnings[a]) or include a tricky connection between sums of the current day and sums of all subsequent days (cumulative earnings). This connection is far more obvious when a graphic representation like a table is used. While algebraic representations can play a role in relation to part C of the problem (daily earnings[b]) and could be used more easily with the other two methods of payment, it appears to be the most complex representational choice. Students experimenting with stating general rules on paper had a difficult time of this as evidenced by: misunderstanding/misstating the patterns; not finding the cumulative earnings; or wanting to make variables out of everything in sight (see samples below). In addition to the difficulty of generating algebraic representations, the amount of effort necessary to generate them appeared to be disproportionate to the leverage they provided in answering the actual question being asked.

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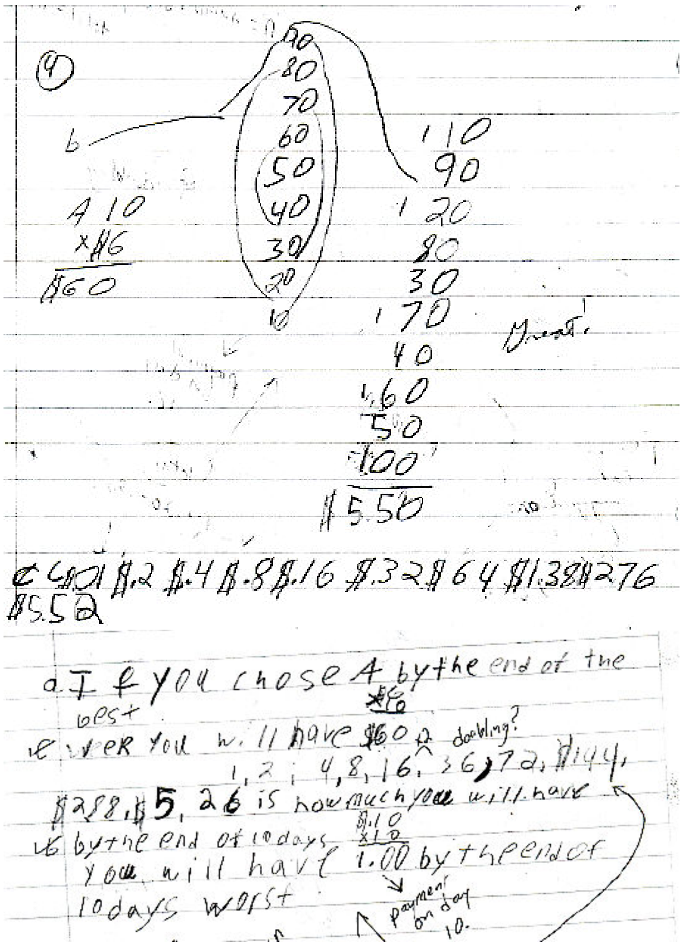

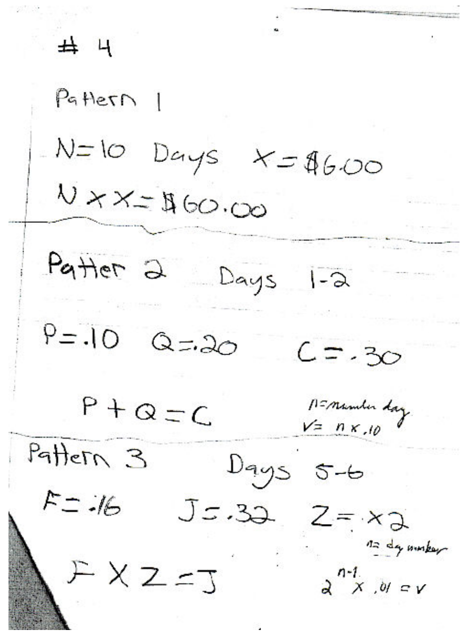

Algebraic Insight

These were samples from early in the year. What I perceived at first as indications of difficulties with algebraic representations were in some cases logical precursors to more sophisticated uses of representations. For instance, when students made everything in sight into variables, perhaps they were recording a sort of first level analysis of what was happening in the problem. It may have been a way to step back from the specifics in the problem, if only a very small amount. This is not where I wanted students to end up, but it provided me more information about how students were thinking about the problem and was evidence that students were stepping, ever so slightly, away from the particulars of numerical methods that they initially preferred.

Graphic Issue:

The menu problem

> Find three consecutive even numbers (like 16, 18, 20) that have a sum of 3,000? Share your reasoning about whether this problem can have more than one solution. Use what you learned in this problem to find three consecutive even numbers that have a sum of 6,000.
>
> Bonus: Find five consecutive even numbers that have a sum of 10,000. Explain in a sentence how you found the answer.

Representations related to the problem solution:

Numerical: 998 + 1,000 + 1,002 = 3,000

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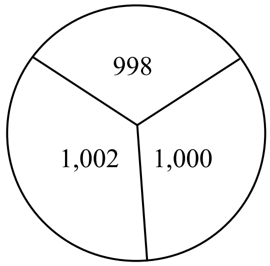
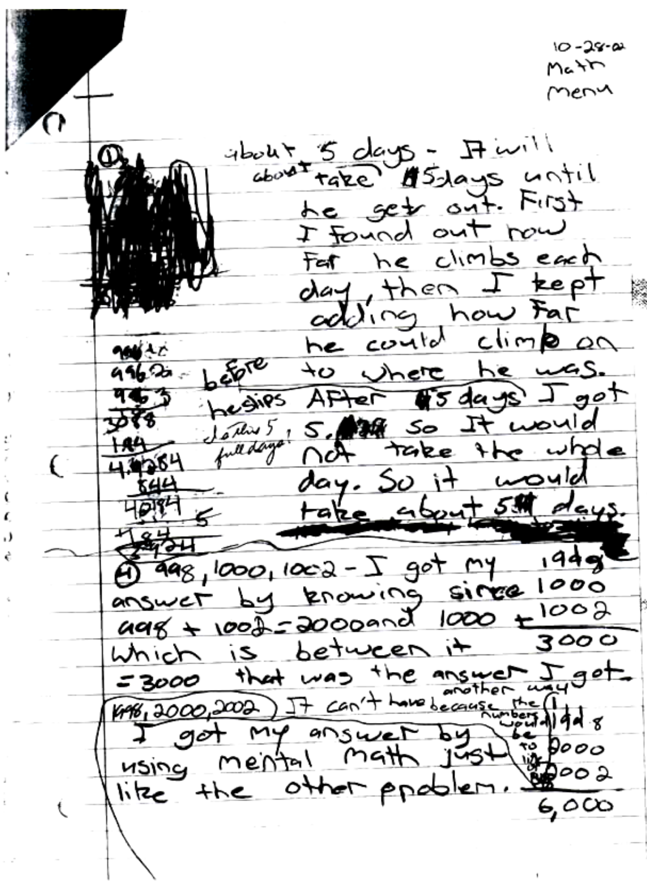
Algebraic: 3,000 ÷ 3= a (a-2) + a + (a+2)= 3,000

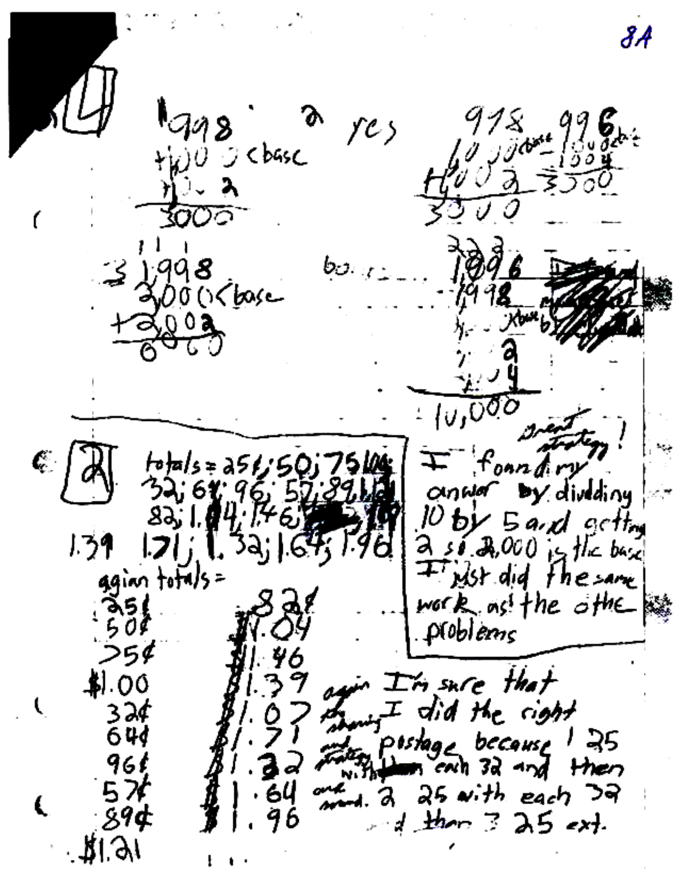
Graphic:



Discussion

Many students utilized numerical representations in a guess and check strategy or recorded numerical representations of their "mental math" work (see #4 in sample on the left below).





Some students also began generalizing, in a written linguistic expression of algebraic thinking, noting that they could find any set of such numbers by dividing the total by the number of terms sought in the problem (see #4 in the sample on the right above). Once that number had been found, then successive additions and subtractions of two would lead to an answer.

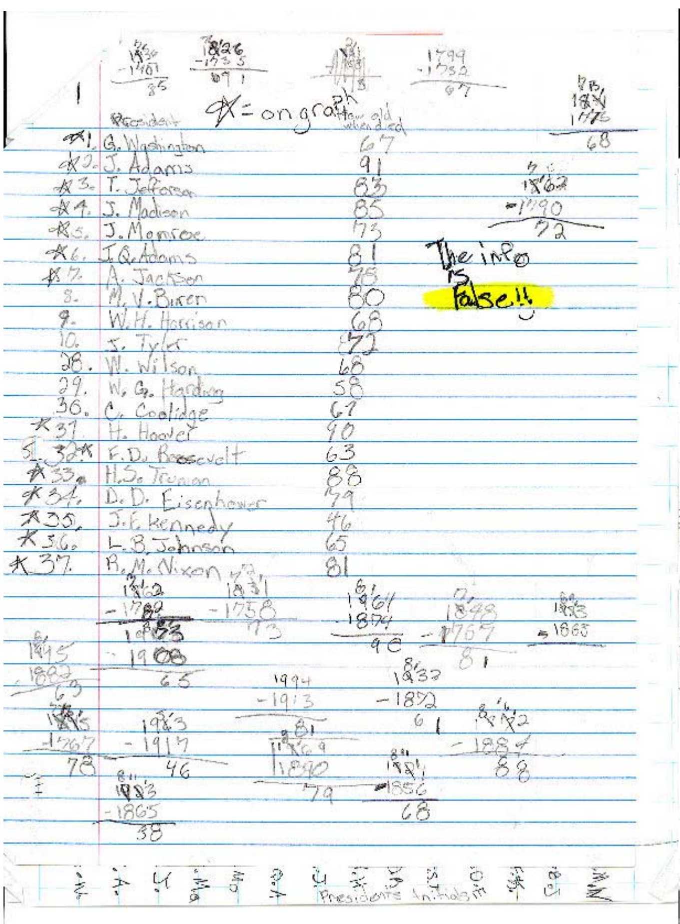
Just as was the case for the algebraic examples in the previous section, some menu problems did not have obvious or particularly useful graphic representations. In this case the first graphic representations that I generated (the circle graph above), seemed better suited to communicating already determined solutions, but not all that useful in determining an initial solution. However, graphic representations come in many forms beyond standard graphs. Whereas random guessing and checking of numbers to arrive at a solution was employed by some students (purely numerical approach), other students spatially organized their numerical work and were systematic in their approach (melding graphic and numerical representations).

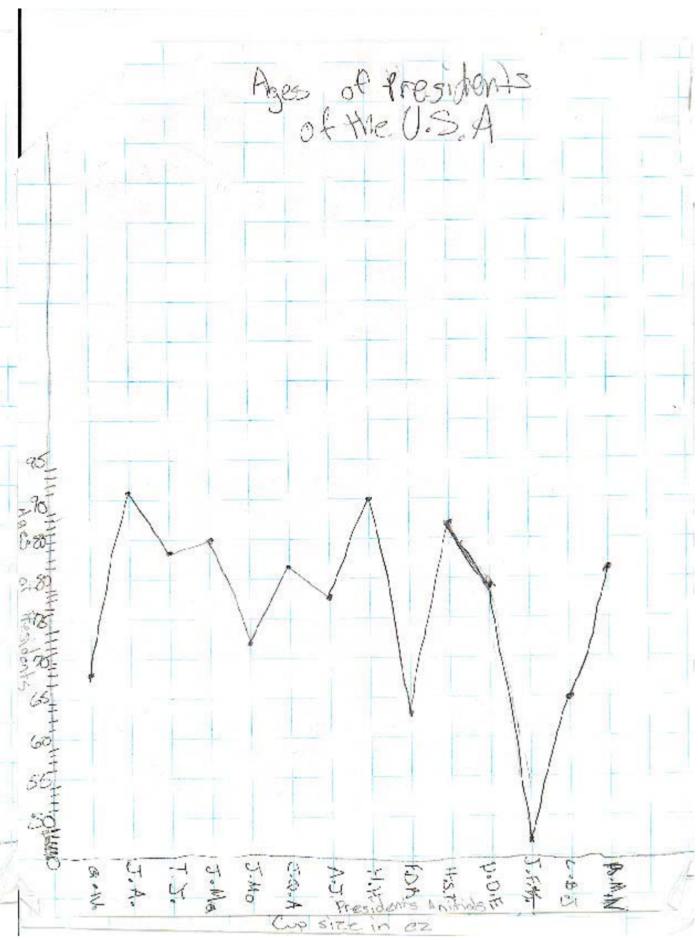
Graphic Insight

Graphic representations were often sites of mathematical experimentation. In some cases the idiosyncratic results of student experimentation caused me to rethink the mathematics involved in menu problems, but other times the representations were used in questionable ways or were so non-standard that they were difficult to understand (for me and peers) without additional written or oral clarification. This leads to a sticky problem illustrated in the following example.

The menu problem

You heard a reporter say that the average age a president lives to be has doubled since the founding of our country. Examine a data chart and state whether you agree or disagree. Express support for your answer by comparing a group of early presidents to a group of recent presidents using numbers and using a graph.





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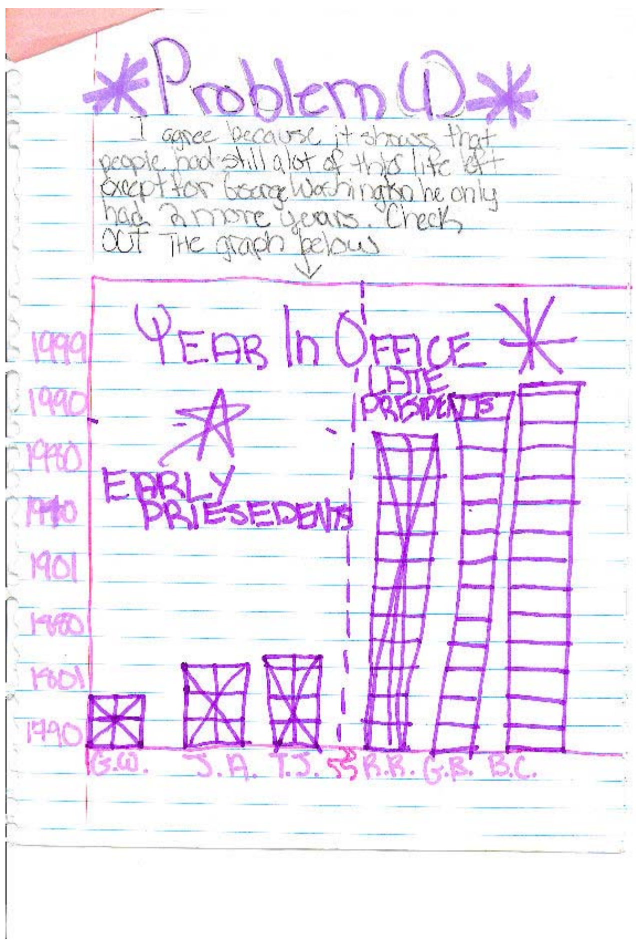
Consider the role of representations in these work samples by imagining that the "using numbers and using a graph" did not appear in the original problem. If representations seen in the work are for the purpose of *supporting ongoing work and sense making* like problem solving, then unusual representations do not necessarily have to be understood or agree upon by others. Individual students would judge the worth of the representation according to the efficiency and accuracy of the problem solving process. Likewise if the purpose of using representations in the work is to *determine or verify the completeness or accuracy of a solution* then the outcome is valued over the clarity of representations used in the process. In the sample above the choice to use a line graph is technically incorrect since the data in the problem is not continuous, however the student's final solution is correct. This graph was constructed to support an answer determined through numerical means. While it does convey the ages when the presidents die, it is unclear as to how the graph actually supports conclusions related to the problem. In comparison the sample below employed a bar graph that is more consistent with the type of data in the problem, but it contains information that did not help the student solve the problem under consideration.



Since teachers are charged with helping students to grow and learn more sophisticated approaches over time, it is also necessary for teachers (and even peers) to gain access to the representations of individual students (this is certainly the case in the problem above since the representations are being used to support solutions). In other words, if a representation does not fulfill the purpose of *communicating ideas to others*, it may diminish or decrease learning opportunities and in many cases impact perceptions of proficiency. While there are other purposes for representations, three important ones appear to be their use in supporting ongoing mathematical work, determining solutions, and communication. While it would be nice if representations served all roles equally well all of the time, it appears that this is not the case.

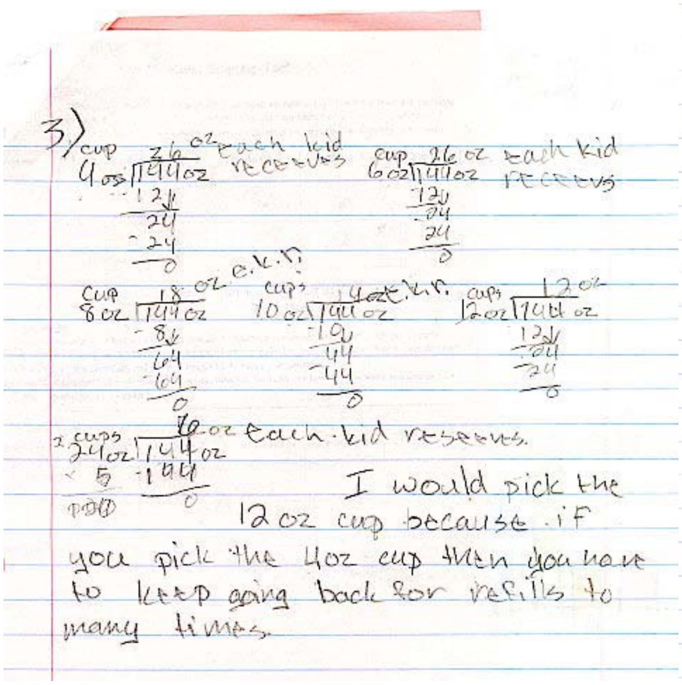
Numerical Issue:

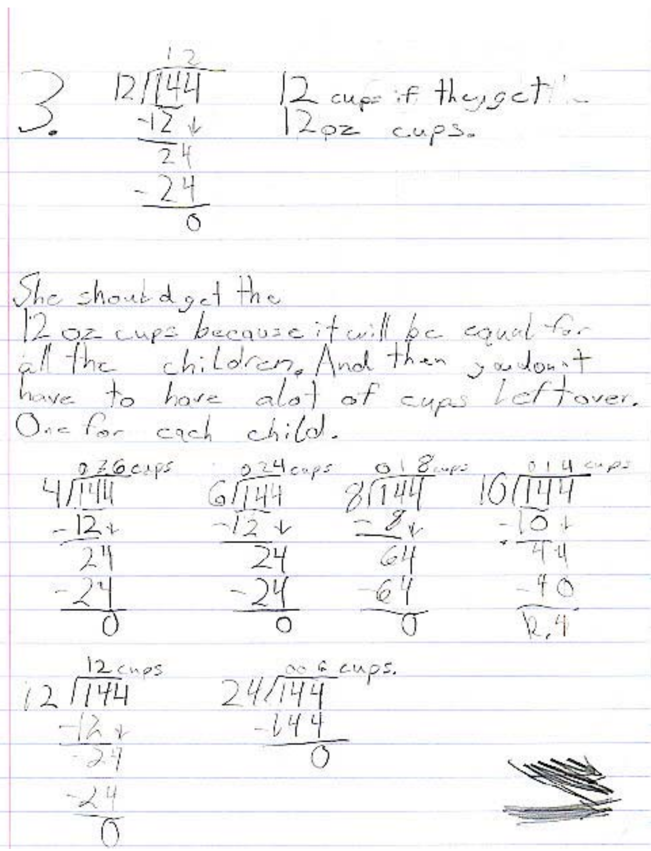
The menu problem

A teacher purchased a GIAGANTIC jug (144 oz) of juice for her 12 students. When she reached the cup isle she was faced with a big choice. There were 4oz, 6oz, 8oz, 10oz, 12oz, and 24 oz cups from which to choose. She wants to make sure that everyone gets the same amount of juice. How many cups could she fill using each of the sizes and the big jug of juice? What size cups should she get and why?

Bonus: Create a graph that shows cup size and the total number of cups poured. What is the shape of the graph?

As was commonly the case, most students used numerical representations to solve the problem.





The incentive to explore graphic representation in the problem only enticed about half of the students to construct a graph. The constructed graphs varied in terms of type, accuracy, and alignment of the graphs with the problem being posed.

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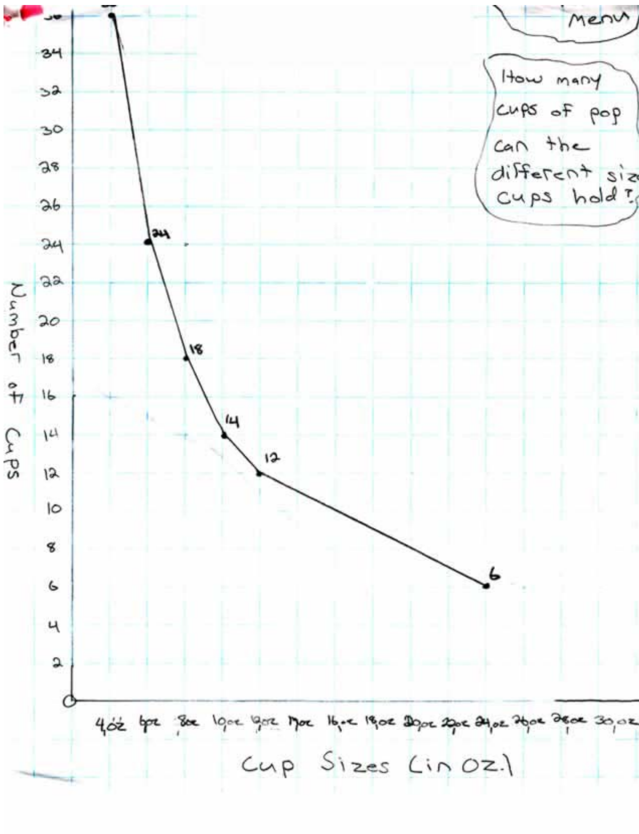
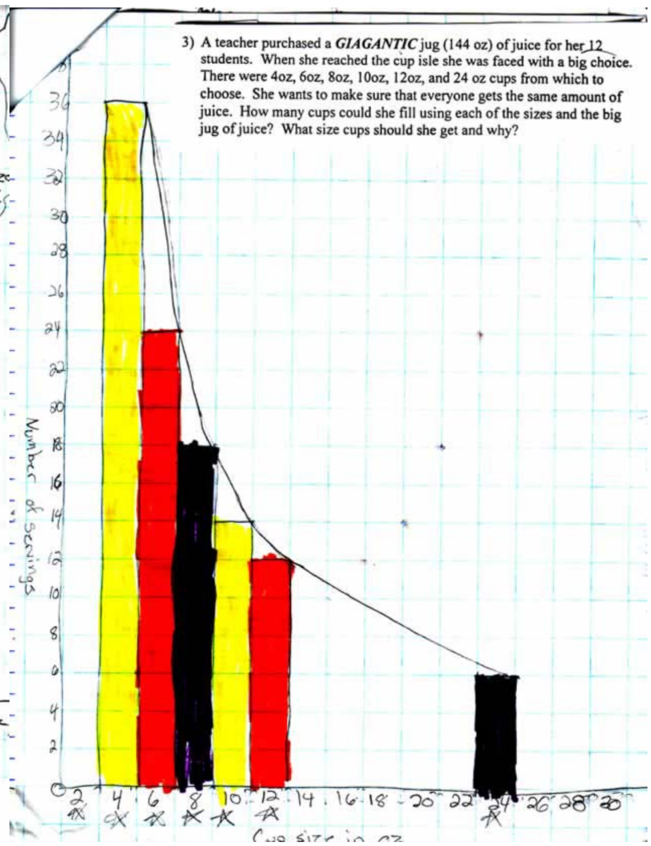
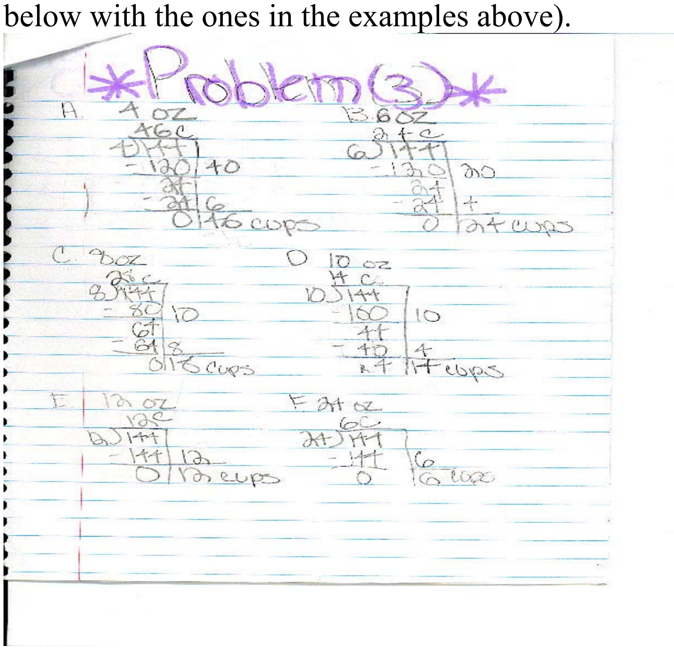
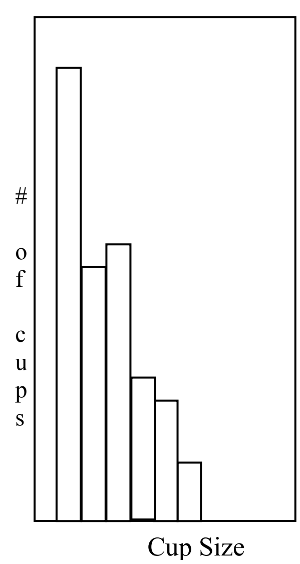




Discussion

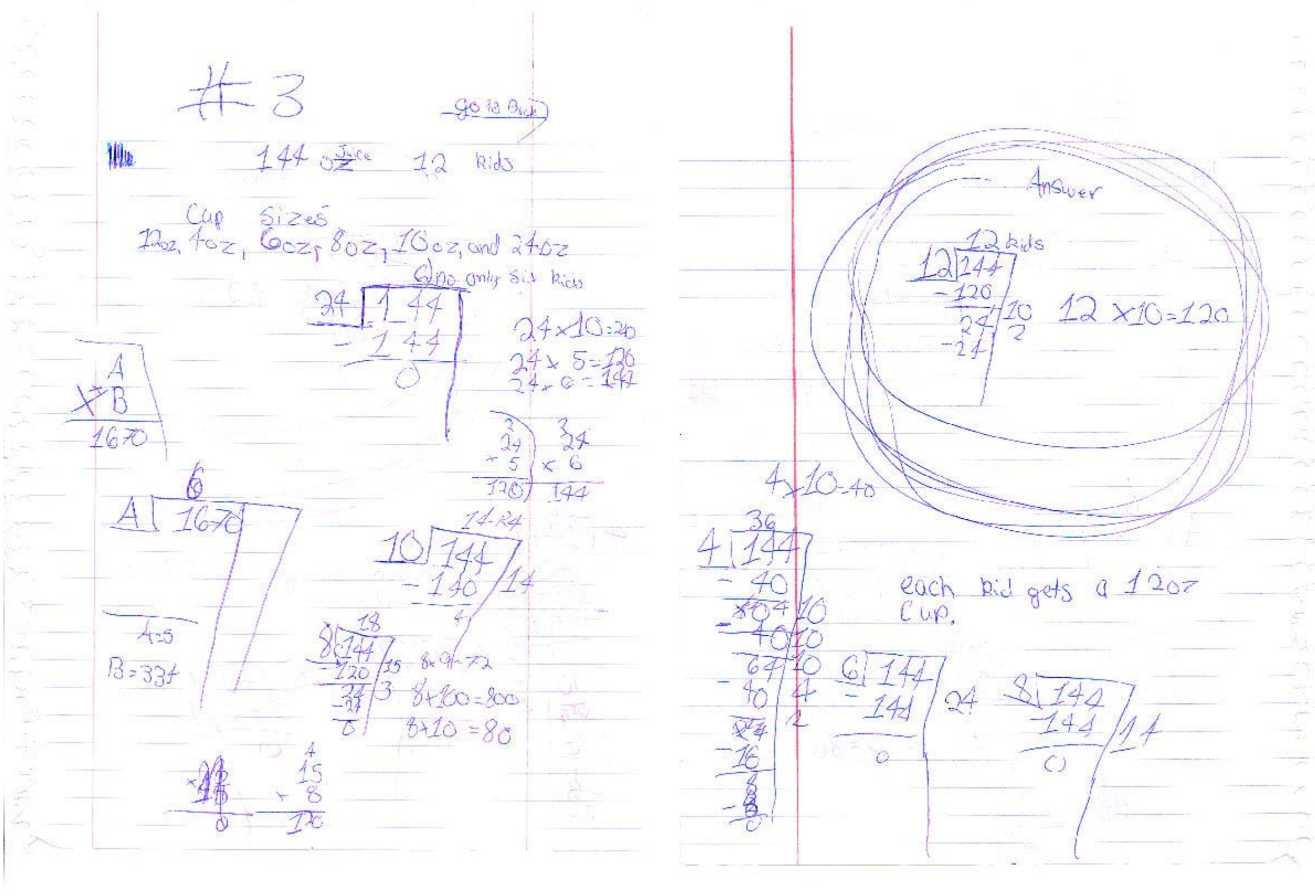
It may be the case that very few students used graphs to solve or convey solutions (no students used algebraic representations) because they did not perceive them as useful tools to accomplish the work involved in the problem. There were however a number of functions that other representations could fulfill in the context of this problem. For instance, a graph might have helped the following student recognize that some of her numerical answers were not sensible (the wrong answers would have disrupted the pattern of the graph as seen when contrasting the graph below with the ones in the examples above).





Certainly the student could have checked her answer through numerical means (multiplying the quotient by the divisor should yield the dividend), but many students were interested and a bit surprised by the idea that a graph could also be useful for this purpose. Of course the graph could only be used for this purpose if students knew how to construct the graph and how to read and interpret what information it portrayed (e.g. students would need to know that something is wrong when an otherwise symmetric graph has a big jump or dip in it).

Another example relates to the potential of an algebraic expression to organize work on the problem. Some students found the numerous calculations required by the problem hard to manage, especially when work needed to be carried over from math to another setting (spare time in class, home or class the next day)



Having a general algebraic expression of the work to be done might have helped in this situation. If students could see, near the outset, that the problem basically required trying different values of C in the expression $144 \div C = N$ (where C= size of the cup in ounces and N= the number of cups that could be filled) then problem solving work would be comprised of comparing resulting values of N to the problem context where 12 students needed to be served. In other problems students liked the "plugging in a value" feature of algebraic representations so it may have been a helpful tool. It is important to keep in mind that the use of an algebraic expression in this case had just as much potential to confuse the situation as it did to support ongoing work. However, the fact that the students and I know that an algebraic expression is an option that may be of use *is* at issue. I needed to build awareness that we did have other representational tools to use depending upon the sorts of challenges we faced.

Numerical Insight

This section may seem decidedly *not* about numerical representations. This is because the biggest issue that I faced in my teaching was that students relied too singularly on numerical representations. In other words, my difficulty was not to calculate ways to involve students with numerical representations, but rather how to nurture a consistent diversity of representational consideration and use. This is not to say that students were always masterful in their use of numbers. Some examples of this from the menu problem considered above include: choosing the wrong numbers or operations to address the problem; flawed execution of algorithms; little evidence checking for accuracy using numerical means. However, I was more worried about the general trend that numbers were used in cases where other representations may have been just as or more useful in undertaking the work related to the problem. Since the terrain of elementary school mathematics is so dominated by work involving specific numbers, it is reasonable to believe that students had built up a broad repertoire of uses for numbers and situational awareness of when they were useful that far surpassed their knowledge of how/when/why to use other representations to solve problems. The tasks that I utilized in class were certainly part of the problem. Many of the menu problems were centered in numerical work or at least are not framed in ways that encouraged or "forced" the consideration of other representations. As the year progressed I did insert more explicit language in the problems directing students to other representations. I also planned whole group discussions that illustrated the uses of graphic/algebraic representations, connected these representations to numerical work, and evaluated the relative strengths of different representations in particular problem contexts.

The Modest Beginnings of Pedagogical Guidelines

From my exploration of the mathematics within the problems that were the basis of teaching and learning in my classroom, it was amazing to see the variety and depth of possibilities presented by each problem. It was also quite humbling to realize that I hadn't perceived those possibilities earlier. The danger at this point is to be content with a "more is better" philosophy of teaching and expect that my students would learn more mathematics by simply opening Pandora's representational box. While there is still a great distance between what I know and what I need to know in order routinely use the Rule of 3 in creating productive learning opportunities for students, the examples and discussion here contain a few guidelines that I keep in mind as I continue my instructional and scholarly work. These ideas may benefit from consolidation and also provide a bit of closure to this piece. Many of the ideas in this conclusion are developed in greater depth in other links available in the reflection section of this website.

The most essential guideline that I try to keep in mind is the *leverage* that different representations supply at particular points in the problem solving process. Leverage involves integrated attention to at least two questions. First, what is the <u>utility</u> of a representation? I need to consider how well the representation supports work on the immediate task (short term) and how well use of the representation will prepare students for doing important representational work on subsequent problems (long term). It is not always possible to determine this ahead of time, as students will encounter mathematical challenges that I do not anticipate or have far different purposes for using the representations. I try to be ready for these obstacles, but keep my eye on the major representational points for whole class purposes. This relates to the second utility question that asks, how accessible is the representation? In some cases very powerful representations are available that will not facilitate problem solving work or communication because students will not understand or be able to employ them. An algebraic expression might elegantly capture what is happening in a problem, but involve the use of exponents that are not yet a part of student thinking[ii]. The developmental distance between where students are and the representation is an important consideration in determining leverage. Being flexible about the form a representation assumes enables greater access. Students can often engage with some forms of representation (particularly algebraic representation) in informal and oral ways before they are able to independently produce them in writing. Creative facilitation of informal oral work may enable the initial engagement of more students with representations that may otherwise seem to be beyond their reach.

Another guideline to consider when thinking about representation in the classroom is the *purpose* for which representations are employed. The immediate purpose for most students is to tackle the math problems that are posed, so a representation must appear to meaningfully or expeditiously serve that end. In the process of problem-based work the three purposes shared in previous examples (support of ongoing work, development and verification of solutions, and communication with others) may be in operation. Different thresholds for judging representational quality apply to these different purposes. A relatively idiosyncratic or underdeveloped representation may be just what is needed when initially solving a problem, but may simultaneously be inadequate for supporting communication. My students and I are routinely aligning representations and purposes. This is a particularly important consideration in the context of trying to help students see the benefits afforded by multiple representations. For instance, I need to uncover the purpose of their representational work when I am circulating around the room as students engage in problem solving. I will question or suggest different representations depending upon the extent that students appear to be working on verifying solutions or on preparing to share thinking with the class. If the fit of representational possibilities I suggest is a chronic mismatch with their purposes there is a potential that students will grow to mistrust, underutilize, or misuse different representations[iii]. It may be difficult for a representation to serve multiple purposes. Sometimes graphs that communicate solutions very well are developed from murky, highly personal numerical experimentation. The later representation cannot exist without the former, but the former cannot accomplish the important work of the later. An idea that follows from attention to this guideline is that it may be possible to increase the extent to which students develop and value multiple representations by expanding the purposes that representations must address.

A final guideline concerns integrated, critical attention to representations. Given the fact that many students are quite experienced with using numerical representations to solve problems, my interactions with students and instructional planning need to orchestrate opportunities to develop conceptual and situational knowledge about the use of graphic and algebraic representations in the problem solving process. I also need to know how and when to point out the strengths and weaknesses of different representations in particular problem contexts. This is where the traditional elementary emphasis of numerical work comes full circle. Being part of the system, I know many of the typical pitfalls, strengths and weaknesses of numerical approaches to problem solving. In comparison I have much less experience with graphic and algebraic approaches, aside from their treatment as discrete instructional topics. I also have less experience in generating or tweaking problems so that graphic/algebraic approaches appear to provide adequate leverage. As a result I need to build up my own understanding of the development of

proficiency with representations, such as algebraic ones, so that I know where to focus my attention when creating tasks and leading discussions. Along the way it will also be important to learn ways to orchestrate discussions so that problem based work with one representation flows into discussion of solutions that involve another representation. Perhaps there are connection points between different representations that will create meaningful coherence.
Over time I hope to instill in students a sense that knowledge and skill with multiple representations amounts to more than mathematical gymnastics, but rather provide tools that were quite useful in solving problems, generalizing to larger situations, and communicating about mathematics with different audiences.

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[i] The Hughes-Hallett et al text (1994) states, "Every topic should be presented geometrically, numerically, and algebraically. We continually encourage our students to think about the geometrical and numerical meaning of what they are doing" (pg. vii). Looking back on this statement it is possible to read it as stressing that multiple representations are used at various points when exploring a *topic*. It is also possible to be persuaded that *continual encouragement* means day to day work with multiple representations. Of course, thoughtful teacher readers of any single claim should exercise caution before proceeding head long into any instructional intervention. However, due to the novelty of this sort of thinking in relation to problem solving instruction, there is also reason to believe that it may be difficult, at least at first, for teachers to know when and where multiple representations might make instructional sense.
[ii] This is not to say that in certain cases it would be unwise to introduce content for student thinking in this way. This also is not an endorsement of the building block approach where student access is determined by possession of a prerequisite number of subskills.
[iii] This is not to say that there are not situations where student purposes, like completing a problem quickly, and my purposes, constructively noticing an error, are at odds. In such cases I am obligated to act on behalf of my own purpose and hopefully help the student see the merits of that purpose in the process.